Averaging Method to the Optimal Control Problem of a Non-Linear Differential Inclusion with Fast-Oscillating Coefficients on Finite Interval

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1 Introduction

The intensive development of science and technology regularly stimulates the search for effective methods for control of various natural, economic, social, and technical processes. Mathematical models of such situations are problems of optimal control of various classes of evolutionary systems. Considerable attention is paid to mathematical models of processes in the form of differential equations and inclusions with a small parameter. For their solution, asymptotic methods are widely used, in particular, the averaging method, the strict mathematical justification of which was proposed by M. M. Krylov and M. M. Bogolyubov. In works of V. A. Plotnikov and works of his school (see, for example, [12]) there is the strict justification of the averaging method in application to control problems.

It is known that the averaging method is one of the most effective tools for solving various optimal control problems for differential equations [4, 5, 8, 9] as well as for differential inclusions with fast oscillating coefficients [6, 7, 13]. The Krasnoselski-Krein theorem [8] and its multi-valued analogue [11] play an essential role for investigation of above-mentioned problems. The concept of integral continuity plays a key role in investigation of the considered optimal control problem using averaging method, since the existence of the limit when we pass to aversged coefficients is equivalent to the integral continuity.

In the present paper we consider the optimal control problem of a non-linear system of differential inclusions with fast oscillating parameters. First, we prove the existence of solutions for the initial perturbed optimal control problem and corresponding problem with averaged coefficients. Then we prove that optimal control of the problem with averaging coefficients can be considered as “approximately” optimal for the initial perturbed one.

2 Setting of the problem and main results

Let us consider an optimal control problem

$$\begin{cases} \dot{x}(t) \in X\Big(\dfrac{t}{\varepsilon}, x(t), u(t)\Big), \quad t \in (0,T), \\ x(0) = x_0, \quad u(\,\cdot\,) \in U, \\ J[x,u] = \displaystyle\int\limits_0^T L(t, x(t), u(t))\,dt + \Phi(x(T)) \to \inf. \end{cases} \tag{2.1}$$

Here $\varepsilon > 0$ is a small parameter, $x : [0,T] \to \mathbb{R}$ is an unknown phase variable, $u : [0,T] \to \mathbb{R}^m$ is an unknown control function, $X : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \operatorname{conv}(\mathbb{R}^n)$ is a multi-valued function, $U \subset L^2(0,T)$ is a fixed set.

Assume that uniformly with respect to x for every $u\in\mathbb{R}^m$

$$\operatorname{dist}_H\bigg(\frac{1}{s}\int\limits_0^s X(\tau,x,u)\,d\tau, Y(x,u)\bigg)\to 0,\ \ s\to\infty, \tag{2.2}$$

where limits for multi-valued function are considered in the sense of [1,3], dist_H is the Hausdorff metric, $Y:\mathbb{R}^n\times\mathbb{R}^m\to\operatorname{conv}(\mathbb{R}^n)$, and the integral of multi-valued function is considered in the sense of Aumann [2]. We consider the following problem with averaged right hand side:

$$\begin{cases} \dot y(t)\in Y(y(t),u(t)),\\ y(0)=x_0,\ \ u(\,\cdot\,)\in U,\\ J[x,u]=\displaystyle\int\limits_0^T L(t,y(t),u(t))\,dt+\Phi(x(T))\to\inf. \end{cases} \tag{2.3}$$

Under the natural assumptions on X, L, Φ, U we will show that problems (2.1) and (2.3) have solutions $\{\overline{x}_\varepsilon,\overline{u}_\varepsilon\}$ and $\{\overline{y},\overline{u}\}$, respectively,

$$\overline{J}_{\varepsilon_n}\to\overline{J},\ \ \varepsilon_n\to 0,$$

where $\overline{J}_{\varepsilon_n}:=J[\overline{x}_{\varepsilon_n},\overline{u}_{\varepsilon_n}]$, $\overline{J}:=[\overline{y},\overline{u}]$, and up to a subsequence

$$\begin{aligned} &\overline{u}_{\varepsilon_n}\to\overline{u}\ \text{ in } L^2(0,T),\\ &\overline{x}_{\varepsilon_n}\to\overline{y}\ \text{ in } \mathbb{C}([0,T]). \end{aligned}$$

In what follows we consider the problem of finding an approximate solution of (2.1) by transition to averaged coefficients. We note that the transition to the averaging parameters can essentially simplify the problem.

Let us consider some assumptions and notations regarding parameters of our problem.

Let $Q=\{t\geq 0,\ x\in\mathbb{R}^n,\ u\in\mathbb{R}^m\}$ and assume the following assumptions hold.

Condition 2.1. Mapping $t,x,u\mapsto X(t,x,u)$ is continuous in Hausdorff metric.

Condition 2.2. Multi-valued function $X(t,x,u)$ satisfies the growth property: $\exists\,M>0$ such that

$$\|X(t,x,u)\|_+\leq M(1+\|x\|)\ \ \forall\,(t,x,u)\in Q,$$

where

$$\|X(t,x,u)\|_+=\sup_{\xi\in X(t,x,u)}\|\xi\|,$$

$\|\xi\|$ is the Euclidian norm of $\xi\in\mathbb{R}^n$;

Condition 2.3. Multi-valued function $X(t,x,u)$ satisfies the Lipschitz condition: $\exists\,\lambda>0$ such that

$$\operatorname{dist}_H\big(X(t,x_1,u_1),X(t,x_2,u_2)\big)\leq\lambda\big(\|x_1-x_2\|+\|u_1-u_2\|\big).$$

Condition 2.4. Mapping $(x,u)\mapsto L(t,x,u)$ is a continuous one, moreover, function $t\mapsto L(t,x,u)$ is measurable $\forall\,x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, and

$$|L(t,x,u)|\leq c(t)\big(1+\|u\|\big),$$

where $c(\,\cdot\,)\in L^2(0,T)$ is a given function.

Condition 2.5. $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a continuous function.

Condition 2.6. $U \subset L^2(0,T)$ is a compact set.

Condition 2.7. Let D and D' be two domains in $\mathbb{R}^n$. We suppose that the embedding $D'+\delta B \subset D$ is fulfilled for some $\delta > 0$, where $B = \{x \in \mathbb{R}^n : \ \|x\| \le 1\}$.

Let us note that under Conditions 2.1–2.3 for all $u \in L^2(0,T)$ the Cauchy problem

$$\begin{cases} \dot{x} \in X\Big(\dfrac{t}{\varepsilon}, x, u\Big), & t \in (0,T) \\ x(0) = x_0 \end{cases} \tag{2.4}$$

has a solution, that is there exists an absolutely continuous function $x : [0,T] \to \mathbb{R}^n$ satisfying the inclusion (2.4) a.e.

Under condition (2.2) the multi-valued mapping Y satisfies the Conditions 2.1–2.3, hence $\forall\, u \in L^2(0,T)$ the Cauchy problem

$$\begin{cases} \dot{y} \in Y(y,u), & t \in (0,T) \\ y(0) = x_0 \end{cases} \tag{2.5}$$

has a solution.

Taking into account conditions for parameters of problem we can show the existence of solutions for the initial perturbed optimal control problem and corresponding problem with averaged coefficients. Namely, we have the next

Theorem 2.1. *Under Conditions 2.1–2.6 problem* (2.1) (*resp. problem* (2.3)) *has the solution* $\{\overline{x}_\varepsilon, \overline{u}_\varepsilon\}$ (*resp.* $\{\overline{y}, \overline{u}\}$).

It worth noting the multi-valued analogue of Krasnoselski–Krein theorem [8,10,11,13] plays an essential role for investigation of the above-mentioned problems. Let us make sure that optimal control of the problem with averaging coefficients can be considered as "approximately" optimal for the initial perturbed one.

Theorem 2.2. *Suppose that for all* $u(\,\cdot\,) \in U$ *problem* (2.5) *has a unique solution. Under Conditions 2.1–2.6 and* (2.2) *we have*

$$\overline{J}_{\varepsilon_n} = J[\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}] \to \overline{J} := J[\overline{y}, \overline{u}] \ \ \text{as} \ \ \varepsilon_n \to 0$$

and up to a subsequence

$$\begin{aligned} &\overline{u}_{\varepsilon_n} \to \overline{u} \ \ \text{in } L^2(0,T), \ \ \varepsilon_n \to 0, \\ &\overline{x}_{\varepsilon_n} \to \overline{y} \ \ \text{in } C(0,T), \ \ \varepsilon_n \to 0, \end{aligned}$$

where $\{\overline{x}_{\varepsilon_n}, \overline{u}_{\varepsilon_n}\}$ *is the solution of* (2.1) *and* $\{\overline{y}, \overline{u}\}$ *is the solution of* (2.3).

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