On the Periodicity of the Riemann Function of Second Order General Type Linear Hyperbolic Equations

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It is well known that the Riemann function $R(\xi,\eta;\xi_1,\eta_1)$ of second order general type linear hyperbolic equations (operators)

$$Lu := u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u = 0$$

is defined by the following way (see, for example, [1–3])

$$L^*R := R_{\xi\eta} - (aR)_\xi - (bR)_\eta + cR = 0, \tag{1}$$

$$R\big|_{\xi=\xi_1} = \exp\bigg\{\int\limits_{\eta_1}^{\eta} a(\xi_1,\eta_2)\,d\eta_2\bigg\}, \quad R\big|_{\eta=\eta_1} = \exp\bigg\{\int\limits_{\xi_1}^{\xi} b(\xi_2,\eta_1)\,d\xi_2\bigg\}. \tag{2}$$

It is also well known that problem $(1),(2)$ is equivalently reduced to the Volterra-type integral equation of the second kind, which, as is well known too, is unconditionally and uniquely solvable for any right-hand side

$$R(\xi,\eta;\xi_1,\eta_1) - \int\limits_{\xi_1}^{\xi} R(\xi_2,\eta;\xi_1,\eta_1)b(\xi_2,\eta)\,d\xi_2 \\ - \int\limits_{\eta_1}^{\eta} R(\xi,\eta_2;\xi_1,\eta_1)a(\xi,\eta_2)\,d\eta_2 + \int\limits_{\xi_1}^{\xi} d\xi_2 \int\limits_{\eta_1}^{\eta} R(\xi_2,\eta_2;\xi_1,\eta_1)c(\xi_2,\eta_2)\,d\eta_2 = 1. \tag{3}$$

In this paper, we will discuss the periodicity of the Riemann function. There is proved the following

Theorem 1. *For the periodicity of the Riemann function in the following sense*

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2) = R(\xi,\eta;\xi_1,\eta_1) \tag{4}$$

it is necessary and sufficient that the following conditions

$$a(\xi+T_1,\eta+T_2) = a(\xi,\eta), \quad b(\xi+T_1,\eta+T_2) = b(\xi,\eta), \quad c(\xi+T_1,\eta+T_2) = c(\xi,\eta) \tag{5}$$

hold.

Proof. Sufficiency. Let us show that along with the function $R(\xi,\eta;\xi_1,\eta_1)$ the solution of the equation (3) is also the function $R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)$, with respect to the variables ξ and η. Consider the following expression

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)-\int\limits_{\xi_1}^{\xi}R(\xi_2+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)b(\xi_2,\eta)\,d\xi_2$$
$$-\int\limits_{\eta_1}^{\eta}R(\xi+T_1,\eta_2+T_2;\xi_1+T_1,\eta_1+T_2)a(\xi,\eta_2)\,d\eta_2$$
$$+\int\limits_{\xi_1}^{\xi}d\xi_2\int\limits_{\eta_1}^{\eta}R(\xi_2+T_1,\eta_2+T_2;\xi_1+T_1,\eta_1+T_2)c(\xi_2,\eta_2)\,d\eta_2.$$

Using transformation of variables ($\xi_2':=\xi_2+T_1$, $\eta_2':=\eta_2+T_2$), the last expression can be rewritten as follows

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)-\int\limits_{\xi_1+T_1}^{\xi+T_1}R(\xi_2',\eta+T_2;\xi_1+T_1,\eta_1+T_2)b(\xi_2'-T_1,\eta)\,d\xi_2'$$
$$-\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi+T_1,\eta_2';\xi_1+T_1,\eta_1+T_2)a(\xi,\eta_2'-T_2)\,d\eta_2'$$
$$+\int\limits_{\xi_1+T_1}^{\xi+T_1}d\xi_2'\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi_2',\eta_2';\xi_1+T_1,\eta_1+T_2)c(\xi_2'-T_1,\eta_2'-T_2)\,d\eta_2',$$

which using the transformation of variables $\xi_2':=\xi_2+T_1$, $\eta_2':=\eta_2+T_2$ can be rewritten as follows

$$R(\xi+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)-\int\limits_{\xi_1+T_1}^{\xi+T_1}R(\xi_2,\eta+T_2;\xi_1+T_1,\eta_1+T_2)b(\xi_2,\eta+T_2)\,d\xi_2$$
$$-\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi+T_1,\eta_2;\xi_1+T_1,\eta_1+T_2)a(\xi+T_1,\eta_2)\,d\eta_2$$
$$+\int\limits_{\xi_1+T_1}^{\xi+T_1}d\xi_2\int\limits_{\eta_1+T_2}^{\eta+T_2}R(\xi_2,\eta_2;\xi_1+T_1,\eta_1+T_2)c(\xi_2,\eta_2)\,d\eta_2=1. \quad (6)$$

From (3) and (6) by virtue of the uniqueness theorem for the solution of the Volterra type integral equation (3), we have that equality (4) is true.

Necessity. From the first equality of (2) and the periodicity condition (4), we obtain

$$\exp\bigg\{\int\limits_{\eta_1}^{\eta}a(\xi_1,\eta_2)\,d\eta_2\bigg\}=R(\xi_1,\eta;\xi_1,\eta_1)=R(\xi_1+T_1,\eta+T_2;\xi_1+T_1,\eta_1+T_2)$$

$$= \exp\left\{\int\limits_{\eta_1+T_2}^{\eta+T_2} a(\xi_1+T_1,\eta_2)\,d\eta_2\right\} = \exp\left\{\int\limits_{\eta_1}^{\eta} a(\xi_1+T_1,\eta_2'+T_2)\,d\eta_2'\right\}$$

and, consequently,

$$\int\limits_{\eta_1}^{\eta} a(\xi_1,\eta_2)\,d\eta_2 = \int\limits_{\eta_1}^{\eta} a(\xi_1+T_1,\eta_2'+T_2)\,d\eta_2'.$$

By differentiating the last equality with respect to the variable η, we get the first equality of (5). Analogously can be obtained the second equality of (5).

Let now obtain the third equality of (5). Indeed, from (1), taking into account the fact that $R(\xi_1,\eta_1;\xi_1,\eta_1)=1$, we have

$$\begin{aligned} c(\xi_1,\eta_1) = -R_{\xi\eta}(\xi_1,\eta_1;\xi_1,\eta_1) + a(\xi_1,\eta_1)R_\xi(\xi_1,\eta_1;\xi_1,\eta_1) \\ + a_\xi(\xi_1,\eta_1) + b(\xi_1,\eta_1)R_\eta(\xi_1,\eta_1;\xi_1,\eta_1) + b_\eta(\xi_1,\eta_1). \end{aligned} \tag{7}$$

Further, from (2) we obtain

$$R_\xi(\xi_1,\eta_1;\xi_1,\eta_1) = b(\xi_1,\eta_1), \quad R_\eta(\xi_1,\eta_1;\xi_1,\eta_1) = a(\xi_1,\eta_1). \tag{8}$$

Considering equalities (7) and (8), we get

$$c(\xi_1,\eta_1) = -R_{\xi\eta}(\xi_1,\eta_1;\xi_1,\eta_1) + 2a(\xi_1,\eta_1)b(\xi_1,\eta_1) + a_\xi(\xi_1,\eta_1) + b_\eta(\xi_1,\eta_1).$$

Therefore, due to the first and second of (5) and (4) equalities, the third equality of (5) is obtained. □

References

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