Anti-Perron Effect of Changing All Positive Characteristic Exponents to Negative in the Linear Case

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We consider the linear differential systems

$$\dot{y} = A(t)y + Q(t)y, \quad y \in \mathbb{R}^n, \quad t \geq t_0, \tag{1_n}$$

with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ which are the first approximation for perturbed linear systems

$$\dot{y} = A(t)y + Q(t)y, \quad y \in \mathbb{R}^n, \quad t \geq t_0, \tag{2_n}$$

and also with infinitely differentiable $n \times n$-matrices $Q(t)$.

O. Perron [7] (see also [6, pp. 50–51]) established in the two-dimensional case the existence of systems (1_2) with exponents $\lambda_1(A) \leq \lambda_2(A) < 0$ and with an infinitely differentiable vector function

$$f(t,y) : (t,y) \in [t_0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}^2,$$

satisfying the condition

$$\|f(t,y)\| \leq C_f \|y\|^m, \quad y \in \mathbb{R}^2, \quad t \geq t_0, \tag{4}$$

for $m = 2$ such that all nontrivial solutions of the perturbed system

$$\dot{y} = A(t)y + f(t,y), \quad y \in \mathbb{R}^2, \quad t \geq t_0 \tag{5}$$

are infinitely extendable to the right, and their Lyapunov exponents form the set $\{\lambda_2(A), \lambda\}$ with some number $\lambda > 0$. This effect of changing the negative exponents of linear approximation (1_2) to positive ones for solutions of the perturbed system (5) with an m-perturbation (4) of an arbitrary order $m > 1$ was studied in a series of our works, including those with S. K. Korovin, and ended (see [2, 3]) with a complete description of Suslin's sets of collections $\Lambda_+(A, f)$ and $\Lambda_-(A, f)$, respectively, of the positive and negative exponents of all nontrivial solutions of system (4), including the necessary case $\Lambda_-(A, f) = \varnothing$.

For possible applications (dealing with the transformation of "absolutely unstable" differential systems into exponentially or conditionally stable ones), of greater interest is the opposite anti-Perron effect (6) of changing by small perturbations (linear, both vanishing at infinity and exponentially decreasing; nonlinear of higher order of smallness) all positive characteristic exponents of linear approximation (1_n) into negative ones for the solutions of the perturbed system. In [4], this effect is investigated for exponentially decreasing linear perturbations: it is proved that the linear systems (1_n) with all positive exponents and the perturbed system (2_n) with an infinitely differentiable $n \times n$-matrix $Q(t)$ satisfying the condition

$$\|Q(t)\| \le C_Q e^{-\sigma t}, \ \ \sigma > 0, \ \ t \ge t_0, \tag{6}$$

and with the characteristic exponents

$$\lambda_1(A+Q) \le \cdots \le \lambda_{n-1}(A+Q) < 0 < \lambda_n(A+Q) \tag{7}$$

exist.

At the same time, the question formulated in this paper on the existence of system (2_n) with perturbation (6) and with a negative higher exponent $\lambda_n(A+Q)$, remains open. Is it possible under a more general perturbation $Q(t) \to 0$, $t \to +\infty$ to realize simultaneously all the necessary inequalities $\lambda_i(A) > 0$, $\lambda_i(A+Q) < 0$, $i = \overline{1,n}$?

An affirmative answer contains the following

Theorem. *For any parameters*

$$\lambda_n \ge \cdots \ge \lambda_1 > 0, \quad \mu_1 \le \cdots \le \mu_n < 0, \ \ 2 \le n \in \mathbb{N},$$

there exist:

1) *a linear system* (1_n) *with bounded infinitely differentiable coefficients and characteristic exponents* $\lambda_i(A) = \lambda_i$, $i = \overline{1,n}$*;*

2) *an infinitely differentiable* $n \times n$*-matrix* $Q(t) \to 0$ *as* $t \to +\infty$ *such that the perturbed system* (2_n) *has characteristic exponents* $\lambda_i(A+Q) = \mu_i$, $i = \overline{1,n}$.

The *proof* of this theorem reduces to the proofs of its two particular variants, respectively, in two-dimensional and three-dimensional cases. In addition, just as in [4], first of all, we construct a piecewise constant and bounded in the interval $[t_0, +\infty)$ matrix $A(t)$ of coefficients of system (1_n) with exponents $\lambda_i(A) = \lambda_i$, $i = \overline{1,n}$, and also the necessary piecewise constant $n \times n$-perturbation matrix $Q(t) \to 0$, $t \to +\infty$ such that the perturbed system (2_n) has characteristic exponents

$$\lambda_i(A+Q) = \mu_i, \ \ i = \overline{1,n}.$$

Next, using the corresponding Gelbaum–Olmsted functions [1, p. 54], we redefine the matrices $A(t)$ and $Q(t)$ in the intervals of very small length containing their discontinuity points in such a way that they become infinitely differentiable and still remain bounded on the semi-axis $[t_0, +\infty)$ (as in the Perron effect itself), while retaining [5] the values of the original and perturbed systems.

References

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