Asymptotic Representations of Rapid Varying Solutions of Differential Equations Asymptotically Close to the Equations with Regularly Varying Nonlinearities

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The differential equation

$$y^n = f(t, y, y', \dots, y^{n-1}) \tag{1}$$

is considered. Here $n \geq 2$, $f : [\alpha, \omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \times \cdots \times \Delta_{Y_{n-1}} \to \mathbb{R}$ is some continuous function, $-\infty < \alpha < \omega \leq +\infty$, Y_j equals to zero, or to $+\infty$, Δ_{Y_j} is some one-sided neighborhood of Y_j, $j = 0, 1, \dots, n-1$.

The asymptotic estimations for singular, quickly varying, and Kneser solutions of equation (1) are described in the monograph by I. T. Kiguradze, T. A. Chanturia [4].

Definition 1. The solution y of equation (1), defined on the interval $[t_0, \omega[\, \subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if the next conditions take place

$$y^{(j)}(t) \in \Delta_{Y_j} \ \text{ as } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(j)}(t) = Y_j \ \ (j = 0, 1, \dots, n-1), \quad \lim_{t \uparrow \omega)} \frac{[y^{n-1}(t)]^2}{y^{n-2}(t) y^n(t)} = \lambda_0.$$

The asymptotic behavior of such solutions earlier has been investigated in the works by V. M. Evtukhov and A. M. Klopot [1–3, 5] for the differential equation

$$y^n = \sum_{n-1}^{m} a_i p_i(t) \prod_{j=0}^{n-1} \varphi_{ij}(y^{(j)}),$$

where $n \geq 2$, $\alpha_i \in \{-1; 1\}$, $p_i : [\alpha, \omega[\, \to \,]0, +\infty[$ is a continuous function $i = 1, \dots, m$, $-\infty < \alpha < \omega \leq +\infty$, $\varphi_{ij} : \Delta_{Y_j} \to \,]0, +\infty[$ is a continuous regularly varying as $y^{(j)} \to Y_j$ function of order

$$\sigma_j, \ \ j = 0, 1, \dots, n-1 \ (i - 1, \dots, m).$$

The aim of the paper is to establish the necessary and sufficient conditions of the existence of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solutions of equation (1) and to find the asymptotic representations of such solutions and their derivatives to the order $n-1$ including.

Every $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solution of the differential equation (1) has (see, for example, [1]) the next a priori asymptotic properties

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \cdots \sim \frac{y^n(t)}{y^{n-1}(t)} \ \text{ as } \ t \uparrow \omega, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) y'(t)}{y(t)} = \pm\infty,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \ \omega = +\infty, \\ t - \omega & \text{if } \ \omega < +\infty. \end{cases}$$

Definition 2. The function f in the differential equation (1) is called a function, that satisfies the condition $(RN)_1$, if there exist a number $\alpha_0 \in \{-1;1\}$, a continuous function $p : [\alpha,\omega[\to]0,+\infty[$ continuous varying as $z \to Y_j$ $(j = 0,1,\dots,n-1)$, functions $\varphi_j : \Delta_{Y_j} \to]0,+\infty[$ $(j = 0,1,\dots,n-1)$ of orders σ_j $(j = 0,1,\dots,n-1)$, such that for all continuously differentiable functions $z_j : [\alpha,\omega[\to \Delta_{Y_j}$ $(j = 0,1,\dots,n-1)$, satisfying the conditions

$$\lim_{t\uparrow\omega} z_j(t) = Y_j, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) z_j'(t)}{z_j(t)} = \pm\infty \ \ (j = 0,1,\dots,n-1),$$
$$\lim_{t\uparrow\omega} \frac{z_{n-1}'(t) z_j(t)}{z_{n-1}(t) z_j'(t)} = 1 \ \ (j = 1,\dots,n-1),$$

the next representation takes place

$$f(t, z_0(t), z_1(t), \dots, z_{n-1}(t)) = \alpha_0 p(t) \prod_{j=0}^{n-1} \varphi_j(z_j(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

Furthermore, we will use the following notations.

$$\gamma = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-2} \sigma_j (n-j-1),$$

$$\nu_j = \begin{cases} 1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta_{Y_j} \text{ is the right neighborhood of zero,} \\ -1 & \text{if } Y_j = +\infty, \text{ or } Y_j = 0 \text{ and } \Delta_{Y_j} \text{ is the left neighborhood of zero} \end{cases}$$
$$(j = 0,1,\dots,n-1),$$

$$J_0(t) = \int_{A_0}^{t} p(s)\,ds, \quad J_{00}(t) = \int_{A_{00}}^{t} J_0(s)\,ds,$$

where

$$A_0 = \begin{cases} \alpha & \text{if } \int_\alpha^\omega p(s)\,ds = +\infty, \\ \omega & \text{if } \int_\alpha^\omega p(s)\,ds < +\infty, \end{cases} \qquad A_{00} = \begin{cases} \alpha & \text{if } \int_\alpha^\omega |J_0(s)|\,ds = +\infty, \\ \omega & \text{if } \int_\alpha^\omega |J_0(s)|\,ds < +\infty. \end{cases}$$

Theorem. *Let the function f satisfy the condition $(RN)_1$ and $\gamma \neq 0$. Then for the existence of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, 1)$-solutions of equation* (1) *the next conditions are necessary:*

$$\frac{p(t)}{J_0(t)} \sim \frac{J_0(t)}{J_{00}(t)} \ \ \text{as} \ \ t\uparrow\omega,$$
$$\lim_{t\uparrow\omega} \frac{\pi_w(t) p(t)}{J_0(t)} = \pm\infty, \quad \nu_j \lim_{t\uparrow\omega} |J_0(t)|^{1/\gamma} = Y_j \ \ (j = 0,1,\dots,n-1),$$

and for $t \in]\alpha,\omega[$, the next inequalities take place

$$\alpha_0 V_{n-1} \gamma J_0(t) > 0, \ \ \nu_j \nu_{n-1} (\gamma J_0(t))^{n-j-1} > 0 \ \ (j = 0,1,\dots,n-2).$$

As the algebraic p equation

$$(1+p)^n = \sum_{j=0}^{n-1} \sigma_j (1+p^j \tag{2}$$

has no roots with zero real part, the conditions are also sufficient for the existence of such solutions of equation (1). *Moreover, for such solutions the next asymptotic representations*

$$y^j(t) = \Big(\frac{\gamma J_{00}(t)}{J_0(t)}\Big)^{n-j-1} y^{n-1}(t)[1+o(1)] \ \ (j=0,1,\dots,n-2), \tag{3}$$

$$\frac{|y^{(n-1)}(t)|^\gamma}{\prod\limits_{j=0}^{n-1} L_j\big(\frac{\gamma J_{00}(t)}{J_0(t)}\big)^{n-j-1} y^{n-1}(t)} = \gamma J_0(t)\Big|\frac{\gamma J_{00}(t)}{J_0(t)}\Big|^{\mu_n}[1+0(1)] \tag{4}$$

take place as $t \uparrow \omega$. *Here*

$$L_j(y^{(j)}) = |y^{(j)}|^{-\sigma_j}\varphi_j(y^{(j)}(t)) \ \ (j=0,1,\dots,n-1).$$

There exists m*-parametric family of such solutions, if among the roots of equation* (2) *there exists* m *roots (taking into account multiple roots), the real parts of which have the sign that is among opposite to the sign of* $\alpha_0 V_{n-1}$.

References

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