Asymptotic Representation for Unbounded Solutions of Higher Order Differential Equations

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Consider the higher order nonlinear equation

$$u^{(n)} + q(t)u^{(n-2)} + r(t)|u|^{\lambda} \operatorname{sgn} u = 0, \quad n \geq 3, \tag{1}$$

where the functions r and q are continuous for $t \geq 1$, q is positive and $\lambda > 0$.

We study equation (1) as a perturbation of the linear differential equation

$$y^{(n)} + q(t)y^{(n-2)} = 0, \quad n \geq 3. \tag{2}$$

Some contributions on the proximity of solutions of two differential equations can be found in the quoted monograph [9], in the papers [1–3, 5] and references therein, in which this problem has been studied in various directions for a large variety of equations. Here we present a survey on some results concerning this topic, which are obtained by the authors and others in the last ten years, see [2–5].

An important role on this problem is played by the second order linear equation

$$h'' + q(t)h = 0. \tag{3}$$

Prototypes of (3) are equations with $q(t) \equiv 1$ and $q(t) \equiv 0$. When $q(t) \equiv 1$, then (3) is oscillatory and this case has been considered in [8]. More precisely, in [8] it was shown that, if r is positive and sufficient large in some sense, then for n even every proper solution of

$$u^{(n)} + u^{(n-2)} + r(t)|u|^{\lambda} \operatorname{sgn} u = 0 \tag{4}$$

is oscillatory, and for n odd every proper solution of (4) is oscillatory, or is vanishing at infinity together with its derivatives, or admits the asymptotic representation

$$x(t) = c(1 + \sin(t - \varphi)) + \varepsilon(t),$$

where c, φ are suitable constants and ε is a continuous function for $t \geq 0$ which vanishes at infinity. According to [8], such equation is said to have *property* A', see also [9] for more details.

On the other hand, if $q(t) \equiv 0$, then (3) is nonoscillatory. This case has been studied in [7], where it is proved that if $-r(t) = \varrho(t) > 0$ is sufficient small in some sense, then the equation

$$u^{(n)} = \varrho(t)|u|^{\lambda} \operatorname{sgn} u, \quad \lambda > 1, \tag{5}$$

has an $(n-1)$ parametric family of so-called rapidly increasing solutions, satisfying the condition

$$\lim_{t\to\infty} |u^{(n-1)}(t)| = \infty,$$

see also [9] for more details.

When (3) is oscillatory, the asymptotic representation of solutions to (1) has been studied by authors in [3, 5] and the main results have been summarized in [6]. Here, we continue such a study by considering the opposite case, that is the case in which (3) is nonoscillatory. Using some results from [2, Theorem 1], we obtain the following

Theorem 1. *Let the second order differential equation* (3) *be nonoscillatory and*

$$\int_1^\infty tq(t)\,dt = \infty. \tag{6}$$

Assume that for some real number $m \in [0, n-1]$,

$$\int_1^\infty t^{n+m\lambda}\,|r(t)|\,dt < \infty. \tag{7}$$

Then for any solution y *to* (2) *such that* $y(t) = O(t^m)$, *there exists a solution* u *to (1) such that for large* t

$$u^{(i)}(t) = y^{(i)}(t) + \varepsilon_i(t), \quad i = 0, 1, \ldots, n-1, \tag{8}$$

where all ε_i *are functions of bounded variation and* $\lim_{t\to\infty} \varepsilon_i(t) = 0$.

The proof is based on the induction method, an iterative process and suitable estimates for solutions to (2). A similar approach has been used in [3], but using completely different estimations for solutions of (2).

Now consider the special case of (1), i.e. the equation

$$u^{(n)}(t) + \frac{\sigma}{t^2}\,u^{(n-2)}(t) + r(t)|u|^\lambda \operatorname{sgn} u = 0, \quad n \ge 3, \tag{9}$$

where $\sigma \in (0, 1/4)$. Obviously, (6) is satisfied and the corresponding second order equation is the Euler equation

$$h''(t) + \frac{\sigma}{t^2}\,h(t) = 0,$$

which is nonoscillatory and whose solutions are known, see, e.g. [10, p. 45]. Using suitable estimations for solutions of (2), we have the following theorem see [2, Corollary 3].

Theorem 2. *Let* $\sigma \in (0, 1/4)$ *and assume that*

$$\int_1^\infty t^{n-1+\gamma\lambda}\,|r(t)|\,dt < \infty,$$

where

$$\gamma = n - 2^{-1}\big(3 + \sqrt{1-4\sigma}\,\big).$$

Then for any polynomial Q *with* $\deg Q \le n-3$, *there exist solutions* u *of* (9) *such that for large* t

$$u^{(i)}(t) = \big(c_1\Gamma_1(t) + c_2\Gamma_2(t) + Q(t)\big)^{(i)} + \varepsilon_i(t), \quad i = 0, \ldots, n-1,$$

where

$$\Gamma_1(t)=\int\limits_1^t(t-s)^{n-3}s^{\mu}\,ds\,O(t^{\beta}),\quad \Gamma_2(t)=\int\limits_1^t(t-s)^{n-3}s^{v}\,ds\,O(t^{\gamma}),$$

$$\mu=2^{-1}\big(1-\sqrt{1-4\sigma}\,\big),\quad \nu=2^{-1}\big(1+\sqrt{1-4\sigma}\,\big),$$

c_1, c_2 *are constants and functions* ε_i *are of bounded variation for large* t *and* $\lim\limits_{t\to\infty}\varepsilon_i(t)=0$.

The following example illustrates Theorem 1.

Example 1. Let $\lambda>0$ and consider the nonlinear equation for $t\geq 1$

$$u^{(4)}+\frac{1}{t^2\log et}\,u^{(2)}=\frac{e^{-t}(t^2\log et+1)}{(1+e^{-t})^{\lambda}t^2\log et}\,|u|^{\lambda}\operatorname{sgn}u. \tag{10}$$

A solution of (10) is

$$u(t)=t+e^{-t}. \tag{11}$$

Setting

$$q(t)=\frac{1}{t^2\log et}\,,\quad r(t)=\frac{e^{-t}(t^2\log et+1)}{(1+e^{-t})^{\lambda}t^2\log et}\,,$$

we get that (3) is nonoscillatory and (6) is valid. Moreover, we have for any $\sigma>0$

$$\int\limits_1^{\infty}t^{\sigma}r(t)\,dt<\infty.$$

Thus, all the assumptions of Theorem 1 are verified with $m=1$ and so equation (10) has a solution u such that for any large t

$$u^{(i)}(t)=y^{(i)}(t)+\varepsilon_i(t),\quad i=0,1,2,3,$$

where ε_i are functions of bounded variation such that $\lim\limits_{t\to\infty}\varepsilon_i(t)=0$ and $y(t)=t$, as the solution (11) illustrates.

Finally, consider the fourth-order differential equation with deviating argument

$$x^{(4)}(t)+q(t)x''(t)+r(t)|x(\varphi(t))|^{\lambda}\operatorname{sgn}x(\varphi(t))=0,\quad \lambda>0, \tag{12}$$

where φ is a nonnegative continuous function for $t\geq 1$ and $\varphi(1)=1$, $\lim\limits_{t\to\infty}\varphi(t)=\infty$. From [3, Theorem 1], if q is a continuously differentiable bounded away from zero function, i.e. $q(t)\geq q_0>0$ for large t, such that

$$\int\limits_1^{\infty}|q'(t)|\,dt<\infty, \tag{13}$$

and

$$\int\limits_1^{\infty}t^{\lambda+1}|r(t)|\,dt<\infty, \tag{14}$$

then (12) with $\varphi(t)=t$ has a solution x such that

$$x^{(i)}(t)=t^i+\varepsilon_i(t),\quad i=0,1,2,3,$$

where functions ε_i are of bounded variation for large t and $\lim_{t\to\infty} \varepsilon_i(t) = 0$. In [4] this result has been improved for a more general equation than (12), without the assumption $\varphi(t) = t$. More precisely, by means of a topological method jointly with certain integral inequalities, the following asymptotic representation of unbounded solutions of (12) has been given, see [4, Corollary 4.1].

Theorem 3. *Let $r(t) \neq 0$ for large t. If q is a continuously differentiable bounded away from zero function satisfying* (13)*, then* (12) *has an asymptotic linear solution x, i.e. a solution x satisfying*

$$\lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} x'(t) = c_x \neq 0, \tag{15}$$

if and only if

$$\int_{t_0}^{\infty} |r(t)|\varphi^{\lambda}(t)\, dt < \infty. \tag{16}$$

Theorem 3 illustrates the dependence of asymptotic linear solutions from the behavior of the deviating argument φ as $t \to \infty$. Moreover, in view of (14) and (16), when $\varphi(t) = t$, Theorem 2 improves the quoted result in [3, Theorem 1]. The following example illustrates this fact.

Example 2. Consider the equation

$$x^{(4)}(t) + x''(t) + \frac{1}{(t+1)^2}\, |x(t^{1/2})|^{3/2} \operatorname{sgn} x(t^{1/2}) = 0, \ \ t \geq 1. \tag{17}$$

By Theorem 3 equation (17) has unbounded asymptotic linear solutions. On the other hand, the corresponding equation

$$x^{(4)}(t) + x''(t) + \frac{1}{(t+1)^2}\, |x(t)|^{3/2} \operatorname{sgn} x(t) = 0, \ \ t \geq 1, \tag{18}$$

does not have solutions x satisfying (15). Indeed, by contradiction, let x be an eventually positive solution x of (18) satisfying (15). Since we have for some $T \geq 1$

$$\int_{T}^{\infty} \frac{x^{3/2}(t)}{(t+1)^2}\, dt = \infty,$$

from (18) we get

$$\lim_{t\to\infty} (x'''(t) + x'(t)) = -\infty,$$

which gives a contradiction with (15).

Since the function q considered in Theorem 3 is bounded away from zero, the corresponding second order equation (3) is oscillatory. Thus, in view of the above mentioned result for equation (5), it is natural to ask under which assumptions on deviating argument φ the above results continue to hold for (12) or, more generally, for (1) when q is small so that (3) is nonoscillatory.

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