# On the Solution of Nonlinear Boundary Value Problems with a Small Parameter in a Special Critical Case

**Sergey Chuiko**

*Donbass State Pedagogical University, Slavyansk, Ukraine*
*E-mail:* chujko-slav@ukr.net

**Nikita Popov**

*Institute of Applied Mathematics and Mechanics of National Academy of Sciences of Ukraine Slavyansk, Ukraine*

The study of weakly nonlinear boundary value problems for systems of ordinary differential equations is a traditional direction for the Kyiv school of nonlinear oscillations [3, 11]. A special critical case for such problems occurs when the equation defining the generating solution turns into an identity [4, 11]. Necessary and sufficient conditions for the solvability of weakly nonlinear boundary value problems in a special critical case are found in the work [4].

## 1 Statement of the problem

We study the problem of constructing a solution

$$z(t,\varepsilon): \ \ z(\,\cdot\,,\varepsilon) \in \mathbb{C}^1[a,b], \ \ z(t,\,\cdot\,) \in \mathbb{C}[0,\varepsilon_0]$$

to the boundary value problem [3, 4, 11]

$$\frac{dz}{dt} = A(t)z + f(t) + \varepsilon Z(z,t,\varepsilon), \ \ \ell z(\,\cdot\,,\varepsilon) = \alpha + \varepsilon J(z(\,\cdot\,,\varepsilon),\varepsilon). \tag{1.1}$$

We look for the solution of problem (1.1) in a small neighborhood of the solution of the generating Noetherian ($m \neq n$) boundary value problem

$$\frac{dz_0}{dt} = A(t)z_0 + f(t), \ \ \ell z_0(\,\cdot\,) = \alpha, \ \ \alpha \in \mathbb{R}^m. \tag{1.2}$$

Here $A(t)$ is an $(n\times n)$-dimensional matrix and $f(t)$ is an $n$-dimensional column-vector, the elements of which are real functions continuous on the segment $[a,b]$, $\ell z(\,\cdot\,)$ is a linear bounded vector functional

$$\ell z(\,\cdot\,) : \mathbb{C}[a,b] \to \mathbb{R}^m.$$

Nonlinearities $Z(z,t,\varepsilon)$ and

$$J(z(\,\cdot\,,\varepsilon),\varepsilon) : \mathbb{C}[a,b] \to \mathbb{R}^m$$

of the boundary value problem (1.1) are assumed to be twice continuously differentiable with respect to the unknown $z$ in a small neighborhood of the generating solution and by a small parameter $\varepsilon$ in a small positive neighborhood of zero. In addition, we consider the vector function $Z(z,t,\varepsilon)$ to be continuous with respect to the independent variable $t$ on the segment $[a,b]$. We study the critical case ($P_{Q^*} \neq 0$), and we assume that the condition

$$P_{Q_d^*}\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} = 0 \tag{1.3}$$

is fulfilled. In this case, the generating problem has an $(r = n - n_1)$-parametric family of solutions

$$z_0(t, c_0) = X_r(t)c_0 + G[f(s); \alpha](t), \ \ c_0 \in \mathbb{R}^r.$$

Here $X(t)$ is a normal $(X(a) = I_n)$ fundamental matrix of a homogeneous part of the generating system (1.2), $Q := \ell X(\,\cdot\,)$ is an $(m \times n)$-dimensional matrix,

$$\operatorname{rank} Q = n_1, \ \ X_r(t) = X(t)P_{Q_r},$$

$P_{Q_r}$ is an $(n \times r)$-matrix formed from $r$ linearly independent columns of an $(n \times n)$-orthoprojector matrix

$$P_Q : \mathbb{R}^n \to \mathbb{N}(Q),$$

$P_{Q_d^*}$ is an $(r \times n)$-matrix formed from $r$ linearly independent columns of an orthoprojector

$$P_{Q^*} : \mathbb{R}^m \to \mathbb{N}(Q^*),$$

and

$$G[f(s); \alpha](t) = X(t)Q^+\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} + K[f(s)](t)$$

is a generalized Green operator of the generating boundary value problem,

$$K[f(s)](t) = X(t)\int\limits_a^t X^{-1}(s)f(s)\,ds$$

is Green's operator of the Cauchy problem of the generating system, $Q^+$ is the pseudo-inverse Moore–Penrose matrix [3]. To find the necessary conditions for the existence of solutions

$$z(t, \varepsilon) = z_0(t, c_0) + x(t, \varepsilon)$$

of problem (1.1) in the critical case, the equation for the generating constants

$$F_0(c_0) := P_{Q_d^*}\Big\{J(z_0(\,\cdot\,, c_0), 0) - \ell K\big[Z(z_0(s, c_0), s, 0)\big](\,\cdot\,)\Big\} = 0$$

is traditionally used [3, 4, 8, 11]. Let us consider a less studied case when the equation for the generating constants turns into the identity [2, 4, 11]:

$$F_0(c_0) \equiv 0, \ \ c_0 \in \mathbb{R}^r. \tag{1.4}$$

The boundary value problem (1.1) under condition (1.4) according to I. G. Malkin's classification [11, p. 139] represents a special critical case, since the traditional scheme of analysis and construction of solutions [3, 8] for such problems is not applicable. In articles [2, 5], the equation for generating constants is constructed, which determines the necessary conditions for the existence of solutions to problem (1.1) in the special critical case. The sufficient condition for the existence of solutions to problem (1.1) in a special critical case is the simplicity of the roots of this equation [2, 5]. We have found conditions for the existence of solutions to problem (1.1) in the special critical case in the presence of multiple roots of such an equation [2, 5].

## 2 Equations for generating functions

To find the necessary conditions for the existence of solutions $z(t,\varepsilon)$ to problem (1.1) in a small neighborhood of the solution of generating problem (1.2) in article [7], the following equation is proposed:

$$\mathfrak{F}(c_0(\varepsilon)) := P_{Q_d^*}\Big\{J(z_0(\,\cdot\,, c_0(\varepsilon)), \varepsilon) - \ell K\big[Z(z_0(s, c_0(\varepsilon)), s, \varepsilon)\big](\,\cdot\,)\Big\} = 0.$$

Consider the case when the equation for generating constants turns into an identity:

$$F_0(c_0) \equiv 0, \ \ \mathfrak{F}(c_0(\varepsilon)) \not\equiv 0.$$

The solution of the nonlinear boundary value problem (1.1) in a particularly critical case is naturally sought in the vicinity of the solution

$$z_0(t, c_0(\varepsilon)) = X_r(t)c_0(\varepsilon) + G_1(t, c_0(\varepsilon))$$

of the modified generating boundary value problem

$$\frac{dz_0(t, c_0(\varepsilon))}{dt} = A(t)z_0(t, c_0(\varepsilon)) + f(t) + \varepsilon\, Z\big(z_0(t, c_0(\varepsilon)), t, 0\big), \quad (2.1)$$
$$\ell z_0(\,\cdot\,, c_0(\varepsilon)) = \alpha + \varepsilon\, J\big(z_0(\,\cdot\,, c_0(\varepsilon)), 0\big);$$

here

$$G_1(t, c_0(\varepsilon)) := G\Big[f(t) + \varepsilon\, Z\big(z_0(s, c_0(\varepsilon)), s, 0\big); \alpha + \varepsilon\, J\big(z_0(\,\cdot\,, c_0(\varepsilon)), 0\big)\Big](t).$$

Under condition (1.3), the equality

$$z_0(t, c_0(0)) = z_0(t, c_0)$$

holds, therefore, in the special critical case ($F_0(c_0) \equiv 0$), for any value of $c_0 \in \mathbb{R}^r$, the generating boundary value problem (2.1) is solvable. The necessary and sufficient condition for the solvability of the boundary value problem (1.1) in the special critical case has the form

$$F(c(\varepsilon)) := P_{Q_d^*}\Big\{J\big(z_0(\,\cdot\,, c_0) + x(\,\cdot\,, \varepsilon), \varepsilon\big) - \ell K\big[Z(z_0(s, c_0) + x(s, \varepsilon), s, \varepsilon)\big](\,\cdot\,)\Big\} = 0. \quad (2.2)$$

Directing in equality (2.2)

$$z(t, \varepsilon) \to z_0(t, c_0(\varepsilon))$$

with a fixed value of $\varepsilon$, we obtain the necessary condition for the solvability of boundary value problem (1.1)

$$\mathcal{F}_0(c_0(\varepsilon), \varepsilon) := P_{Q_d^*}\Big\{J(z_0(\,\cdot\,, c_0(\varepsilon)), \varepsilon) - \ell K\big[Z(z_0(s, c_0(\varepsilon)), s, \varepsilon)\big](\,\cdot\,)\Big\} = 0. \quad (2.3)$$

In this way, the following lemma is proved.

**Lemma.** *Suppose that for the boundary value problem* (1.1) *there is a special critical case and condition* (1.3) *of the solvability of the generating problem is used. Let us also assume that in a small neighborhood of the generating solution* $z_0(t, c_0^*(\varepsilon))$ *problem* (1.1) *has a solution*

$$z(t, \varepsilon): \ \ z(\,\cdot\,, \varepsilon) \in \mathbb{C}^1[a, b], \quad z(t, \,\cdot\,) \in \mathbb{C}[0, \varepsilon_0].$$

*Then the vector* $c_0^*(\varepsilon) \in \mathbb{R}^r$ *satisfies equation* (2.3).

Equation (2.3) defines the generating solutions $z_0(t, c_0^*(\varepsilon))$, in the small neighborhood of which the sought solutions of boundary value problem (1.1) for the special critical case can be found. By analogy with weakly nonlinear boundary value problems in critical case [3], equation (2.3) will be called the equation for generating functions of the boundary value problem (1.1) in the special critical case. In contrast to article [7], equation (2.3) is built on the basis of the auxiliary boundary value problem (2.1), and not the original generating boundary value problem (1.2), which will be obtained from this problem at $\varepsilon = 0$. To find the solution $c_0^*(\varepsilon) \in \mathbb{R}^r$ of the nonlinear equation (2.3), the Newton–Kantorovich method can be used [1, 6, 9]. The smoothness of the vector $c_0^*(\varepsilon) \in \mathbb{R}^r$ significantly affects the form of the sought solution of the boundary value problem (1.1).

The proposed scheme of studies of the nonlinear boundary value problem (1.1) for the special critical case is a generalization of the results for boundary-value problems for systems of differential equations [1, 3, 4, 8, 10–12].

## References


[1] O. A. Boĭchuk and S. M. Chuĭko, On the approximate solution of weakly nonlinear boundary value problems by the Newton–Kantorovich method. (Ukrainian) *Nelīnīĭnī Koliv.* **23** (2020), no. 3, 321–331; translation in *J. Math. Sci. (N.Y.)* **261** (2022), no. 2, 228–240.

[2] A. A. Boĭchuk, S. M. Chuĭko and A. S. Chuĭko, Nonautonomous periodic boundary value problems in a special critical case. (Russian) *Nelīnīĭnī Koliv.* **7** (2004), no. 1, 53–66; translation in *Nonlinear Oscil. (N.Y.)* **7** (2004), no. 1, 52–64.

[3] A. A. Boichuk and A. M. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary-Value Problems.* VSP, Utrecht, 2004.

[4] S. M. Chuiko, Fredholm boundary value problem in special critical case. *Dokl. Akad. Nauk Ukraine*, 2007, no. 2, 26–30.

[5] S. M. Chuĭko, A weakly nonlinear boundary value problem in a particular critical case. (Russian) *Ukraïn. Mat. Zh.* **61** (2009), no. 4, 548–562; translation in *Ukrainian Math. J.* **61** (2009), no. 4, 657–673.

[6] S. M. Chuiko, To the generalization of the Newton–Kantorovich theorem. *Visnyk of V. N. Karazin Kharkiv National University, Ser. "Mathematics, Applied Mathematics and Mechanics"* **85** (2017), 62–68.

[7] S. M. Chuĭko, E. V. Chuĭko and I. A. Boĭchuk, On the reduction of a Noetherian boundary-value problem to a first-order critical case. (Russian) *Nelīnīĭnī Koliv.* **17** (2014), no. 2, 281–292; translation in *J. Math. Sci. (N.Y.)* **208** (2015), no. 5, 607–619.

[8] E. A. Grebenikov and Yu. A. Ryabov, *Constructive Methods of Analyzing Nonlinear Systems.* (Russian) Nonlinear Analysis and its Applications "Nauka", Moscow, 1979.

[9] L. V. Kantorovich and G. P. Akilov, Functional analysis. (Russian) Izdat. "Nauka", Moscow, 1977.

[10] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Translated in *J. Soviet Math.* **43** (1988), no. 2, 2259–2339. Itogi Nauki i Tekhniki, *Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.

[11] I. G. Malkin, *Some Problems of the Theory of Nonlinear Oscillations.* (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956.

[12] A. M. Samoĭlenko, S. M. Chuĭko and O. V. Nēsmēlova, Nonlinear boundary value problems unsolved with respect to the derivative. (Ukrainian) *Ukraïn. Mat. Zh.* **72** (2020), no. 8, 1106–1118; translation in *Ukrainian Math. J.* **72** (2021), no. 8, 1280–1293.