Slowly Varying Solutions of Essentially Nonlinear Differential Equations of Second Order

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp(R(|\ln|yy'||)), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ are continuous functions, $Y_i \in \{0, \pm\infty\}$ $(i = 0,1)$, Δ_{Y_i} is a onesided neighborhood of Y_i, every function $\varphi_i(z)$ $(i = 0,1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0 + \sigma_1 \ne 1$, $\sigma_1 \ne 0$, the function $R :]0,+\infty[\to]0,+\infty[$ is continuously differentiable and regularly varying on infinity of the order μ, $0 < \mu < 1$, the derivative function of the function R is monotone, is considered in the work.

Definition. The solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$ if it is defined on $[t_0,\omega[\subset [a,\omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0,1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

A lot of works (see, for example, [2, 3]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equations of the form (1), in which $R \equiv 0$. The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1\}$. The asymptotic properties and necessary and sufficient conditions of existence of such solutions of equation (1) have been received in [1].

The case $\lambda_0 = 0$ is one of cases of the most difficulty because in this cases such solutions are slowly varying functions as $t \uparrow \omega$. Some results about asymptotic properties and existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) in this special case are presented in the work.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0,+\infty[$ satisfies the condition S, if for any continuous differentiable function $L : \Delta_{Y_i} \to]0,+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the next equality

$$\Theta(zL(z)) = \Theta(z)(1+o(1)) \ \text{ is true as } \ z \to Y \ \ (z \in \Delta_Y)$$

holds.

We need the next subsidiary notations

$$\pi_\omega(t)=\begin{cases} t & \text{as } \omega=+\infty,\\ t-\omega & \text{as } \omega<+\infty,\end{cases}\quad \Theta_i(z)=\varphi_i(z)|z|^{-\sigma_i}\ \ (i=0,1),$$

$$I(t)=\alpha_0\int_{A_\omega}^{t} p(\tau)d\tau,\quad A_\omega=\begin{cases} a, & \text{if } \int\limits_a^\omega p(\tau)\,d\tau=+\infty,\\ \omega, & \text{if } \int\limits_a^\omega p(\tau)\,d\tau<+\infty.\end{cases}$$

In the case

$$\lim_{t\uparrow\omega}\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}=Y_1,$$

we put

$$J(t)=\int_{B_\omega}^{t}\Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau,$$

$$B_\omega=\begin{cases} b, & \text{if } \int\limits_b^\omega \Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau=+\infty,\\ \omega, & \text{if } \int\limits_b^\omega \Big|I(\tau)\Theta_1\Big(\frac{\operatorname{sign} y_0^1}{|\pi_\omega(t)|}\Big)\Big|^{\frac{1}{1-\sigma_1}}\,d\tau<+\infty,\end{cases}$$

$$N(t)=\frac{(1-\sigma_1)I(t)|(1-\sigma_1)I(t)\Theta_1(\frac{y_1^0}{|\pi_\omega(t)|})|^{\frac{1}{\sigma_1-1}}}{I'(t)R'(|\ln|\pi_\omega(t)||)}\,.$$

Theorem 1. *Let in equation* (1) *the function $\varphi_1(y')$ satisfy the condition S and the next condition take place*

$$\lim_{t\uparrow\omega}\frac{R(|\ln|\pi_\omega(t)||)J(t)}{\pi_\omega(t)\ln|\pi_\omega(t)|J'(t)}=0. \tag{2}$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$-solutions of equation (1) *the next conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega} y_0^0|J(t)|^{\frac{1-\sigma_1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{J'(t)}{y_1^0|J(t)|}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\sigma_1-1,$$

$$\frac{I(t)}{y_1^0(1-\sigma_1)}>0\ \text{ as }\ t\in\,]a,\omega[\,,\qquad \frac{y_0^0y_1^0(1-\sigma_1)J(t)}{1-\sigma_0-\sigma_1}>0\ \text{ as }\ t\in\,]b,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$:

$$\frac{y(t)}{|\exp(R(|\ln|y(t)y'(t)||))\varphi_0(y(t))|^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,|1-\sigma_1|^{\frac{1}{1-\sigma_1}}J(t)[1+o(1)],$$

$$\frac{y(t)}{y'(t)}=\frac{(1-\sigma_0-\sigma_1)J(t))}{(1-\sigma_1)J'(t)}\,[1+o(1)].$$

Theorem 2. *Let in Theorem 1 condition* (2) *do not hold but the function* φ_1 *satisfy the condition* S*,* p *be a twice continuously differentiable function, and the next condition takes place*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=0.$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1)*, for which there exists a finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$*, the next conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0\exp\big(R(|\ln|\pi_\omega(t)||)\big)^{\frac{\sigma_1-1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{-\alpha_0}{\pi_\omega(t)}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\frac{\sigma_1-1}{\alpha_0},$$
$$\alpha_0y_1^0\pi_\omega(t)<0,\quad \alpha_0(1-\sigma_1)(1-\sigma_0-\sigma_1)y_0^0R'\big(|\ln|\pi_\omega(t)||\big)>0.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$*:*

$$\frac{y(t)}{|\varphi_0(y(t))\exp(R(|\ln|y(t)y'(t)||))|^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}N(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{I'(t)R'(|\ln|\pi_\omega(t)||)}{(1-\sigma_0-\sigma_1)(1-\sigma_1)I(t)}[1+o(1)].$$

Theorem 3. *Let in Theorem 1 conditions* (2) *do not hold but the function* φ_1 *satisfy the condition* S*,* p *be a twice continuously differentiable function, and the next condition takes place*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=M\in R\setminus\{0,1\}.$$

Then for the existence of $P_\omega(Y_0,Y_1,0)$*-solutions of equation* (1)*, for which there exists a finite or infinite limit* $\lim\limits_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}$*, the next conditions are necessary and sufficient*

$$\lim_{t\uparrow\omega}y_0^0\big(\exp(R(|\ln|\pi_\omega(t)||))\big)^{\frac{\sigma_1-1}{1-\sigma_0-\sigma_1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{-\alpha_0}{\pi_\omega(t)}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I'(t)}{I(t)}=\frac{\sigma_1-1}{\alpha_0},$$
$$\alpha_0y_1^0\pi_\omega(t)<0,\quad \alpha_0(1-M)(1-\sigma_1)(1-\sigma_0-\sigma_1)y_0^0R'\big(|\ln|\pi_\omega(t)||\big)>0.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$*:*

$$\frac{y(t)}{|\varphi_0(y(t))\exp(R(|\ln|y(t)y'(t)||))|^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{(1-\sigma_1)(1-M)}N(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{I'(t)R'(|\ln|\pi_\omega(t)||)(1-M)}{(1-\sigma_0-\sigma_1)(1-\sigma_1)I(t)}[1+o(1)].$$

Let consider some more specific class of differential equations of the form (1) and use Theorems 1, 2 and 3. The differential equation

$$y''=mt^{\sigma_1-2}\exp(k\ln^\gamma t)|y|^{\sigma_0}|y'|^{\sigma_1}\exp\big((|\ln|yy'||)^\mu\big)\tag{3}$$

on the interval $[t_0,+\infty[$ $(t_0>0)$, where $m\in]-\infty,0[$, $k\in]0,+\infty[$, $\gamma,\mu\in]0;1[$, $\sigma_0,\sigma_1\in\mathbb{R}$, $\sigma_0+\sigma_1\neq1$, $\sigma_1\neq1$, is the equation of the form (1), where

$$\alpha_0=\operatorname{sign}m=-1,\ \ p(t)=mt^{\sigma_1-2}\exp(k\ln^\gamma t),\ \ \varphi_0=|y|^{\sigma_0},\ \ \varphi_1=|y|^{\sigma_1},\ \ R(z)=z^\mu.$$

This function φ_1 satisfies the condition S. Let consider the case when $\omega=Y_0=Y_1=+\infty$.

Using Theorem 1 we obtain that if $\mu-\gamma<0$, then for the existence $P_{+\infty}(+\infty,+\infty,0)$-solutions of equation (3) the following condition

$$1-\sigma_0-\sigma_1>0 \tag{4}$$

is necessary and sufficient.

Moreover, for each such solution the following asymptotic representations take place as $t\to+\infty$:

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}}\exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1-1}\Big)=\frac{1-\sigma_0-\sigma_1}{\gamma k}\exp\Big(\frac{k\ln^{\gamma}t}{1-\sigma_1}\Big)\ln^{1-\gamma}t[1+o(1)],$$
$$\frac{y(t)}{y'(t)}=\frac{(1-\sigma_0-\sigma_1)\gamma k}{(1-\sigma_1)^2}\,\frac{\ln^{\gamma-1}t}{t}\,[1+o(1)].$$

Let us now consider the case $\mu-\gamma>0$. In this case by Theorem 2 we obtain that for $\mu-\gamma>0$ for existence of $P_{+\infty}(+\infty,+\infty,0)$-solutions to equation (3) condition (4) is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t\to+\infty$:

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}}\exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1-1}\Big)=\frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1)}\exp\Big(\frac{k\ln^{\gamma}t}{1-\sigma_1}\Big)\ln^{1-\mu}t[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{\mu}{\sigma_0+\sigma_1-1}\,t^{\sigma_1-2}\ln^{\gamma-1}t[1+o(1)].$$

Let us now consider the case $\mu=\gamma$. By Theorem 3 we obtain that for existence of $P_{+\infty}(+\infty,+\infty,0)$-solutions to equation (3) condition (4) together with the condition

$$(1-\sigma_1-k)(1-\sigma_1)>0$$

is necessary and sufficient. Moreover, each such solution satisfies the next asymptotic representations as $t\to+\infty$:

$$y^{\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}}\exp\Big(\frac{|\ln|y(t)y'(t)||^{\mu}}{\sigma_1-1}\Big)=\frac{1-\sigma_0-\sigma_1}{\mu(1-\sigma_1-k)}\exp\Big(\frac{k\ln^{\gamma}t}{1-\sigma_1}\Big)\ln^{1-\mu}t[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{\mu(1-\sigma_1-k)}{(\sigma_0+\sigma_1-1)(1-\sigma_1)}\,t^{\sigma_1-2}\ln^{\gamma-1}t[1+o(1)].$$

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