Lyapunov Irregularity Coefficient and Exponential Stability Index of a Linear Parametric System as a Vector Function of the Parameter

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1 Introduction. Statement of the problem

For a given positive integer n, let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \overset{\text{def}}{=} [0, +\infty), \tag{1.1}$$

with piecewise continuous and bounded on the half-line $\mathbb{R}_+$ coefficient matrices $A(\,\cdot\,) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$. In what follows, we identify system (1.1) with its coefficient matrix and hence write $A \in \mathcal{M}_n$. For a system $A \in \mathcal{M}_n$, let $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ denote its Lyapunov exponents [7, p. 567], [5, p. 6], $\mathrm{es}(A)$ its exponential stability index, i.e., the dimension of the linear subspace of solutions to system (1.1) that have negative Lyapunov exponents, and $\sigma_{\mathrm{L}}(A)$ its Lyapunov irregularity coefficient [7, p. 563], [5, p. 10], i.e., the quantity

$$\sigma_{\mathrm{L}}(A) \overset{\text{def}}{=} \sum_{i=1}^{n} \lambda_i(A) - \varliminf_{t\to+\infty} \frac{1}{t} \int_0^t \mathrm{tr} A(\tau)\, d\tau,$$

tr being the trace of a matrix. By virtue of the Lyapunov inequality [7, p. 562], the quantity $\sigma_{\mathrm{L}}(A)$ is nonnegative.

The Lyapunov irregularity coefficient is one of the most important asymptotic characteristics of systems in the class $\mathcal{M}_n$. The condition $\sigma_{\mathrm{L}}(A) = 0$ singles out in $\mathcal{M}_n$ the subclass $\mathcal{R}_n$ of Lyapunov regular systems, historically the first class of systems for which the problem of conditional stability by the first approximation was solved in the affirmative [7, p. 578]. Moreover, this coefficient is used to state sufficient conditions characterizing the response of a system $A \in \mathcal{M}_n$ to both exponentially decaying linear perturbations and higher-order nonlinear perturbations. For example, the Lyapunov exponents of a system $A \in \mathcal{M}_n$ are preserved under linear exponentially decaying perturbations $Q(\,\cdot\,)$, whenever the estimate $\|Q(t)\| \leqslant C\exp(-\sigma t)$, $t \in \mathbb{R}_+$, holds with some constants $C > 0$ and $\sigma > \sigma_{\mathrm{L}}(A)$ [3]. If for a higher-order perturbation $f(t, x)$ ($\|f(t, x)\| \leqslant \mathrm{const}\|x\|^m$, $t \in \mathbb{R}_+$, $m = \mathit{const} > 1$) of a system $A \in \mathcal{M}_n$ its order $m > 1$ satisfies the estimate $(m-1)\lambda_n(A) + \sigma_{\mathrm{L}}(A) < 0$, then the trivial solution of the perturbed system is stable (the Lyapunov–Massera theorem [7, pp. 578–579], [8]).

It was a long-standing conjecture that the Lyapunov exponents of Lyapunov regular systems are invariant under perturbations vanishing at infinity. The conjecture was based essentially on the fundamental result by Lyapunov which claims that if a nonlinear system (with natural restrictions on the right-hand side) has a regular first approximation system and the latter is conditionally exponentially stable, then so is the zero solution of the original system (with the same dimension of the stable manifold and asymptotic exponent) [7, pp. 576–578]. Nevertheless, in the paper [10] R. È. Vinograd provided an example of systems $A, B \in \mathcal{R}_2$ satisfying

$$\lambda_1(A) = \lambda_2(A) = 0, \quad \lambda_1(B) = -1, \quad \lambda_2(B) = 1, \quad \lim_{t\to+\infty} \|A(t) - B(t)\| = 0.$$

From this result it follows, in particular, that the exponential stability index $\operatorname{es}(\,\cdot\,)$ – a function taking exactly $n+1$ values – is not upper semicontinuous even on the set $\mathcal{R}_n$ of Lyapunov regular systems with the topology of uniform convergence of coefficients on the semiaxis.

Let M be a metric space. Consider a family

$$\dot{x} = A(t,\mu)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+, \tag{1.2}$$

of linear differential systems depending on a parameter $\mu \in M$ such that for each $\mu \in M$ the matrix-valued function $A(\,\cdot\,,\mu) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded (for every μ, generally, by a different constant). Therefore, fixing a value of the parameter $\mu \in M$ in family (1.2) we obtain a linear differential system with continuous coefficients bounded on the semiaxis. We denote by $\operatorname{es}(\mu; A)$ its exponential stability index and by $\sigma_{\mathrm{L}}(\mu; A)$ its Lyapunov irregularity coefficient.

It is customary to consider a family of matrix-valued functions $A(\,\cdot\,,\mu)$, $\mu \in M$, under one of the following two natural assumptions: that the family is continuous either **a)** in the compact-open topology, or **b)** in the uniform topology. The condition **a)** is equivalent to the fact that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points from M converges to a point μ_0, then the sequence of functions $A(t,\mu_k)$ of the variable $t \in \mathbb{R}_+$ converges to the function $A(t,\mu_0)$ as $k \to +\infty$ uniformly on each segment $[0,T] \subset \mathbb{R}_+$, while the condition **b)** is equivalent to the fact that this convergence is uniform over the whole semiaxis $\mathbb{R}_+$. Denote the class of families (1.2) that are continuous in the compact-open topology by $\mathcal{C}^n(M)$ and the class of those that are continuous in the uniform topology by $\mathcal{U}^n(M)$. It is clear that a proper inclusion $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$ holds. In the sequel, we will identify families (1.2) with the matrix-valued functions $A(\,\cdot\,,\,\cdot\,)$ defining them and therefore write $A \in \mathcal{C}^n(M)$ or $A \in \mathcal{U}^n(M)$.

Along with the class $\mathcal{U}^n(M)$ we consider its subclass $\mathcal{UZ}^n_{\mathcal{R}}(M)$, which is defined as follows. For a number $n \in \mathbb{N}$ and a metric space M, denote by $\mathcal{Z}_n(M)$ the class of jointly continuous matrix-valued functions $Q(\,\cdot\,,\,\cdot\,) : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ that vanish at infinity uniformly over $\mu \in M$ (the last means that $\sup\limits_{\mu\in M} \|Q(t,\mu)\| \to 0$ as $t \to +\infty$). The class $\mathcal{UZ}^n_{\mathcal{R}}(M)$ comprises families

$$\dot{x} = (B(t) + Q(t,\mu))x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+, \tag{1.3}$$

where $B \in \mathcal{R}_n$ and $Q \in \mathcal{Z}_n(M)$. Denoting the coefficient matrix of family (1.3) by $A(t,\mu)$ and, as above, identifying it with the family itself, we will write $A \in \mathcal{UZ}^n_{\mathcal{R}}(M)$.

Problem. For any $n \in \mathbb{N}$ and metric space M obtain a complete function-theoretic description for each of the function classes:

$$\mathfrak{T}\big[\mathcal{C}^n(M)\big] \stackrel{\text{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,;A), \operatorname{es}(\,\cdot\,;A)\big) : \; A \in \mathcal{C}^n(M)\Big\},$$

$$\mathfrak{T}\big[\mathcal{U}^n(M)\big] \stackrel{\text{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,;A), \operatorname{es}(\,\cdot\,;A)\big) : \; A \in \mathcal{U}^n(M)\Big\},$$

$$\mathfrak{T}\big[\mathcal{UZ}^n_{\mathcal{R}}(M)\big] \stackrel{\text{def}}{=} \Big\{\big(\sigma_{\mathrm{L}}(\,\cdot\,;A), \operatorname{es}(\,\cdot\,;A)\big) : \; A \in \mathcal{UZ}^n_{\mathcal{R}}(M)\Big\}.$$

2 Preceding results

Let us recall that a function $f : M \to \mathbb{R}$ is said [4, pp. 266–267] to be of the class $({}^*, G_\delta)$ if for each $r \in \mathbb{R}$, the preimage $f^{-1}([r, +\infty))$ of the half-interval $[r, +\infty)$ is a G_δ-set of the metric space M. In particular, the class $({}^*, G_\delta)$ is a proper subclass of the second Baire class [4, p. 294]. Recall also that a function $m : M \to \mathbb{R}$ is called a majorant of a function $f : M \to \mathbb{R}$ if $f(x) \le m(x)$ for all $x \in M$.

A complete description of the classes

$$\mathfrak{S}[\mathcal{U}^n(M)] \stackrel{\text{def}}{=} \bigl\{\sigma_{\mathrm{L}}(\,\cdot\,; A) :\ A \in \mathcal{U}^n(M)\bigr\} \ \text{ and } \ \mathfrak{S}[\mathcal{UZ}^n_{\mathcal{R}}(M)] \stackrel{\text{def}}{=} \bigl\{\sigma_{\mathrm{L}}(\,\cdot\,; A) :\ A \in \mathcal{UZ}^n_{\mathcal{R}}(M)\bigr\},$$

i.e., the classes made up of the first elements of pairs in the classes $\mathfrak{T}[\mathcal{U}^n(M)]$ and $\mathfrak{T}[\mathcal{UZ}^n_{\mathcal{R}}(M)]$, respectively, is obtained in the paper [2] and is as follows: the classes $\mathfrak{S}[\mathcal{U}^n(M)]$ and $\mathfrak{S}[\mathcal{UZ}^n_{\mathcal{R}}(M)]$ coincide with one another and consist of functions $M \to \mathbb{R}_+$ of the class $({}^*, G_\delta)$ that have a continuous majorant.

A description of the class $\mathfrak{S}[\mathcal{C}^n(M)] \stackrel{\text{def}}{=} \{\sigma_{\mathrm{L}}(\,\cdot\,; A) :\ A \in \mathcal{C}^n(M)\}$ follows from the result of the paper [8]: for any $n \in \mathbb{N}$ and metric space M the class $\mathfrak{S}[\mathcal{C}^n(M)]$ consists of all functions $M \to \mathbb{R}_+$ of the class $({}^*, G_\delta)$. This description can also be immediately drawn from a more general result obtained in the paper [11], which is a complete description of the class $\{(\sigma_{\mathrm{L}}(\,\cdot\,; A), \sigma_{\mathrm{P}}(\,\cdot\,; A)) :\ A \in \mathcal{C}^n(M)\}$ of vector functions composed of the Lyapunov irregularity coefficient σ_{L} and the Perron one σ_{P} [2, p. 10] for families in $\mathcal{C}^n(M)$: for any $n \ge 2$ and metric space M a vector function $(\sigma_1, \sigma_2) : M \to \mathbb{R}^2_+$ belongs to the above mentioned class if and only if the functions σ_1 and σ_2 are $({}^*, G_\delta)$ and for all $\mu \in M$, the inequalities $0 \leqslant \sigma_2(\mu) \leqslant \sigma_1(\mu) \leqslant n\sigma_2(\mu)$ hold. (Recall that the Perron irregularity coefficient $\sigma_{\mathrm{P}}(A)$ of a system $A \in \mathcal{M}_n$ is defined by the equality

$$\sigma_{\mathrm{P}}(A) \stackrel{\text{def}}{=} \max_{1 \le i \le n} \bigl\{\lambda_i(A) + \lambda_{n-i+1}(-A^T)\bigr\};$$

$\sigma_{\mathrm{P}}(\,\cdot\,; A)$ stands for the Perron irregularity coefficient of family (1.2).)

A description of the classes $\{\mathrm{es}(\,\cdot\,; A) :\ A \in \mathcal{C}^n(M)\}$ and $\{\mathrm{es}(\,\cdot\,; A) :\ A \in \mathcal{U}^n(M)\}$ is obtained in the paper [1]: both classes consist of functions $f : M \to \{0, \dots, n\}$ such that the function $(-f)$ is of the class $({}^*, G_\delta)$.

3 The main result

Theorem 3.1. *For any $n \ge 1$ and metric space M a pair of functions (σ, s), where $\sigma : M \to \mathbb{R}_+$ and $s : M \to \{0, \dots, n\}$, belongs to the class $\mathfrak{T}[\mathcal{C}^n(M)]$ if and only if the functions σ and $(-s)$ are of the class $({}^*, G_\delta)$.*

Unfortunately, the authors of the report failed to completely solve the above stated problem on description of the classes $\mathfrak{T}[\mathcal{U}^n(M)]$ and $\mathfrak{T}[\mathcal{UZ}^n_{\mathcal{R}}(M)]$. Below we consider a simplified version of the problem.

Following the report [9], which treats an analogous quantity, we call the *indicator of total exponential instability* of system (1.1) the quantity $\mathrm{ti}(A)$ defined by

$$\mathrm{ti}\,(A) = \begin{cases} 1, & \text{if } \lambda_1(A) \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

The next theorem completely describes the classes of pairs of functions

$$\mathfrak{U}\bigl[\mathcal{U}^n(M)\bigr] \stackrel{\text{def}}{=} \Bigl\{\bigl(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{ti}(\,\cdot\,; A)\bigr) :\ A \in \mathcal{U}^n(M)\Bigr\},$$
$$\mathfrak{U}\bigl[\mathcal{UZ}^n_{\mathcal{R}}(M)\bigr] \stackrel{\text{def}}{=} \Bigl\{\bigl(\sigma_{\mathrm{L}}(\,\cdot\,; A), \mathrm{ti}(\,\cdot\,; A)\bigr) :\ A \in \mathcal{UZ}^n_{\mathcal{R}}(M)\Bigr\}.$$

Theorem 3.2. *For any $n \geq 2$ and metric space M the equality $\mathfrak{U}[\mathcal{U}^n(M)] = \mathfrak{U}[\mathcal{U}\mathcal{Z}^n_{\mathcal{R}}(M)]$ is valid. A pair of functions (σ, t), where $\sigma : M \to \mathbb{R}_+$ and $t : M \to \{0,1\}$, belongs to the above defined classes if and only if the functions σ and t are of the class $({}^*, G_\delta)$ and the function σ has a continuous majorant.*

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