On the Criterion of Well-Posedness of the Cauchy Problem with Weight for Systems of Linear Ordinary Differential Equations with Singularities

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Let $I=[a,b]\subset\mathbb{R}$ be a finite and closed interval non-degenerate in the point, $t_0\in\,]a,b[$ and $I_{t_0}=[a,b]\setminus\{t_0\}$, $I_{t_0}^-=[a,t_0[\,$, $I_{t_0}^+=\,]t_0,b]$.

Consider the Cauchy problem with weight for linear system of ordinary differential equations with singularities

$$\frac{dx}{dt}=P(t)\,x+q(t)\ \text{ for a.a. } t\in I_{t_0}, \tag{1}$$

$$\lim_{t\to t_0}(\Phi^{-1}(t)\,x(t))=0, \tag{2}$$

where $P\in L_{loc}(I_{t_0},\mathbb{R}^{n\times n})$, $q\in L_{loc}(I_{t_0},\mathbb{R}^n)$; $\Phi=\operatorname{diag}(\varphi_1,\dots,\varphi_n)$ is a positive diagonal $n\times n$-matrix valued function defined, continuous on $[a,b]\setminus\{t_0\}$ and having an inverse $\Phi^{-1}(t)$ for $t\in[a,b]\setminus\{t_0\}$.

Along with system (1) consider the sequence of singular systems

$$\frac{dx}{dt}=P_m(t)\,x+q_m(t) \tag{3}$$

$(m=1,2,\dots)$ under condition (2), where P_m and q_m are, as above, a matrix- and vector-functions, respectively.

We discuss the question whether the unique solvability of problem (1), (2) guarantees the unique solvability of problem (3), (2) for each sufficiently large m and nearness of its solutions to the solution of problem (1), (2) in the definite sense if matrix-functions P and P_m and vector-functions q and q_m are nearly among themselves.

The singularity of system (1) is considered in the sense that the matrix P and vector q functions, in general, are not integrable at the point t_0. In general, the solution of problem (1), (2) is not continuous at the point t_0 and, therefore, it can not be a solution in the classical sense. But its restriction on every interval from I_{t_0} is a solution of system (1). In connection with this we remind the following example from [6, 7].

Let $\alpha>0$ and $\varepsilon\in\,]0,\alpha[\,$. Then $x(t)=|t|^{\varepsilon-\alpha}\operatorname{sgn}t$ is the unique solution of the problem

$$\frac{dx}{dt}=-\frac{\alpha\,x}{t}+\varepsilon|t|^{\varepsilon-1-\alpha},\quad \lim_{t\to0}(t^\alpha x(t))=0.$$

The function x is not a solution of the equation on the set $I = \mathbb{R}$, however x is a solution to the above equation only on $\mathbb{R} \setminus \{0\}$.

First, the same problem for the differential system (3) have been investigated by I. Kiguradze (see, [6, 7]), where only the sufficient conditions are obtained. As to sufficient conditions of well-possedness for functional-differential case they are obtained in [8] (see also the references therein).

The necessary and sufficient conditions of well-posedness of problem (1), (2) has been obtained in [1, 2] for the regular case.

To our knowledge, the question on necessary and sufficient conditions of well-posedness of problem (1), (2) with singularity has not been considered up to now. So the problem is actual.

As to the existence of solutions and related problems for system (1), first, they are investigated by V. A. Chechik in the monograph [5]. Similar problems for impulsive differential and so called generalized ordinary differential systems are investigated in [3, 4] and for functional-differential case in [8] (see also the references therein).

We present necessary and sufficient conditions, as well effective sufficient conditions for so called Φ-well-posedness of problem (1), (2).

Throughout the paper we use the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$; $[a, b]$ and $]a, b[$ $(a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n\times m}$ is the space of real $n \times m$ matrices X with the standard norm $\|X\|$.

If $X = (x_{ik})_{i,k=1}^{n,m} \in \mathbb{R}^{n\times m}$, then $|X| = (|x_{ik}|)_{i,k=1}^{n,m}$.

$[X]_\pm = \dfrac{1}{2}\,(|X| \pm X)$, $\mathbb{R}_+^{n\times m} = \big\{(x_{ik})_{i,k=1}^{n,m} : \ x_{ik} \geq 0 \ (i = 1, \dots, n, \ k = 1, \dots, m)\big\}$.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all column n-vectors $x = (x_i)_{i=1}^n$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$.

$O_{n\times m}$ (or O) is the zero $n \times m$-matrix, 0_n (or 0) is the zero n-vector.

I_n is identity $n \times n$-matrix.

If $X \in \mathbb{R}^{n\times n}$, then X^{-1}, $\det X$ and $r(X)$ are, respectively, the matrix inverse to X, the determinant of X and the spectral radius of X.

The inequalities between the matrices are understood componentwisely.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$\mathrm{AC}([a, b]; D)$, where $D \subset \mathbb{R}^{n\times m}$, is the set of all absolutely continuous matrix-functions $X : [a, b] \to D$.

$\mathrm{AC}_{loc}(J; D)$, where $J \subset \mathbb{R}$, is the set of all matrix-functions $X : J \to D$ whose restrictions to an arbitrary closed interval $[a, b] \subset J$ belong to $\mathrm{AC}([a, b]; D)$.

$L_{loc}(I_{t_0}; \mathbb{R}^{n\times m})$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions on every closed interval $[a, b]$ from I_{t_0} belong to $L([a, b]; \mathbb{R}^{n\times m})$.

Under a solution of problem (1), (2) we understand a vector-function $x \in \mathrm{AC}(I_{t_0}; \mathbb{R}^n)$ satisfying the equality $x'(t) = P(t)\,x(t) + q(t)$ for a.a. $t \in I_{t_0}$ and condition (2).

Let $P_* = (p_{*ik})_{i,k=1}^n \in L_{loc}(I_{t_0}; \mathbb{R}^{n\times n})$. A matrix-function $C_* : I_{t_0} \times I_{t_0} \to \mathbb{R}^{n\times n}$ is said to be the Cauchy matrix of the homogeneous system

$$\frac{dx}{dt} = P_*(t)x, \tag{4}$$

if, for each interval $J \subset I$ and $\tau \in J$, the restriction of the matrix-function $C_*(\,\cdot\,, \tau) : I_{t_0} \to \mathbb{R}^{n\times n}$ on J is the fundamental matrix of system (4), satisfying the condition $C_*(\tau, \tau) = I_n$. Therefore, C_* is the Cauchy matrix of system (4) if and only if the restriction of C_* on $J \times J$ is the Cauchy matrix of the system in the regular case.

Definition 1. Problem $(1),(2)$ is said to be Φ-well-posed with respect to the matrix-function P_* if it has a unique solution x_0 and for every sequences of matrix P_m and vector q_m $(m=1,2,\dots)$ functions such that

$$\lim_{t\to t_0}\bigg(\Phi^{-1}(t)\int_{t_0}^{t}(q_m(s)-q(s))\,ds\bigg)=0_n \tag{5}$$

for each sufficiently large m, and conditions

$$\lim_{m\to+\infty}\bigg\|\int_{t_0}^{t}\Phi^{-1}(s)\,|P_m(s)-P(s)|\,\Phi(s)\,ds\bigg\|=0 \ \text{ uniformly on } I \tag{6}$$

and

$$\lim_{m\to+\infty}\bigg(\bigg\|\Phi^{-1}(t)\int_{t_0}^{t}(q_m(s)-q(s))\,ds\bigg\|+\bigg\|\int_{t_0}^{t}\Phi^{-1}(s)\,\bigg|P_*(s)\int_{t_0}^{s}(q_m(\tau)-q(\tau))\,d\tau\bigg|\,ds\bigg\|\bigg)=0 \tag{7}$$

uniformly on I

hold, problem $(3),(2)$ has the unique solution x_m for each sufficiently large m and the condition

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)\,(x_m(t)-x_0(t))\big\|=0 \ \text{ uniformly on } I \tag{8}$$

hold.

The introduced definition is more general than the one given in [6, 7].

Definition 2. We say that the sequence (P_m,q_m) $(m=1,2,\dots)$ belongs to the set $\mathcal{S}_{P_*}(P,q;\Phi,t_0)$ if problem $(3),(2)$ has a unique solution x_m for each sufficiently large m and condition (8) holds.

Theorem 1. *Let there exist a matrix-function $P_*\in L_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ and constant matrices $B_0,B\in\mathbb{R}_+^{n\times n}$ such that*

$$r(B)<1, \tag{9}$$

and the estimates

$$|C_*(t,\tau)|\le\Phi(t)B_0\Phi^{-1}(\tau) \ \text{ for } \ t\in I_{t_0}(\delta),\ \ (t-t_0)(\tau-t_0)>0,\ \ |\tau-t_0|\le|t-t_0| \tag{10}$$

and

$$\bigg|\int_{t_0}^{t}\big|C_*(t,s)(P(s)-P_*(s))\big|\,\Phi(s)\,ds\bigg|\le\Phi(t)B \ \text{ for } \ t\in I_{t_0}(\delta) \tag{11}$$

are fulfilled for some $\delta>0$, where C_* is the Cauchy matrix of system (4). Let, moreover,

$$\lim_{t\to t_0}\bigg\|\int_{t_0}^{t}\Phi^{-1}(t)\,C_*(t,s)q(s)\,ds\bigg\|=0. \tag{12}$$

Then problem $(1),(2)$ is Φ-well-posed with respect to P_*.

Remark 1. Under the conditions of Theorem 1 problem $(1),(2)$ is uniquely solvable (see, [6, 7]). In addition, condition (9) is essential and it cannot be neglected by $r(B)\le 1$, i.e., in the last case the problem may not be solvable. Corresponding example one can find in [6, 7], as well.

Theorem 2. *Let conditions of Theorem 1 be fulfilled and sequences P_m and q_m $(m=1,2,\dots)$ be such that conditions* (6) *and* (7) *hold. Then*

$$((P_m,q_m))_{m=1}^{+\infty}\in\mathcal{S}_{P_*}(P,q;\Phi,t_0). \tag{13}$$

Theorem 3. *Let conditions of Theorem 1 be fulfilled and let there exist a sequence of the non-degenerated matrix-functions $H_m\in \mathrm{AC}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that*

$$\lim_{t\to t_0}\bigg(\Phi^{-1}(t)\int\limits_{t_0}^{t}(q_{m*}(s)-q(s))\,ds\bigg)=0_n \tag{14}$$

for each sufficiently large m. Let, moreover, the conditions

$$\lim_{m\to+\infty}\big\|\Phi^{-1}(t)H_m^{-1}(t)\Phi(t)-I_n\big\|=0, \tag{15}$$

$$\lim_{m\to+\infty}\bigg\|\int\limits_{t_0}^{t}\Phi^{-1}(s)|P_{m*}(s)-P(s)|\Phi(s)\,ds\bigg\|=0 \tag{16}$$

and

$$\lim_{m\to+\infty}\bigg\|\Phi^{-1}(t)\bigg|\int\limits_{t_0}^{t}(q_{m*}(s)-q(s))\,ds\bigg|+\bigg|\int\limits_{t_0}^{t}\Phi^{-1}(s)\bigg|P_*(s)\int\limits_{t_0}^{s}(q_{m*}(\tau)-q(\tau))\,d\tau\bigg|\,ds\bigg|\bigg\|=0 \tag{17}$$

be fulfilled uniformly on I_{t_0}, where $P_{m*}(t)\equiv(H_m'(t)+H_m(t)P_m(t))H_m^{-1}(t)$ and $q_{m*}(t)\equiv H_m(t)q_m(t)$ $(m=1,2,\dots)$. Then inclusion (13) *holds.*

Theorem 4. *Let conditions of Theorem 1 be fulfilled. Let, moreover,*

$$\|B_0\|\,\|(I_n-B)^{-1}\|<1 \tag{18}$$

and

$$\limsup_{t\to t_0}\bigg\|\int\limits_{t_0}^{t}\Phi^{-1}(s)|P_*(s)|\Phi(s)\,ds\bigg\|<+\infty. \tag{19}$$

Then inclusion (13) *holds if and only if there exist a sequence of matrix-functions $H_m\in \mathrm{AC}_{loc}(I_{t_0};\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$ such that condition* (15) *holds uniformly on I, and conditions* (16) *and* (17) *hold uniformly on I_{t_0}, where $P_{m*}(t)$ and q_{m*} $(m=1,2,\dots)$ are defined as in Theorem 3.*

Theorem 4′. *Let conditions of Theorem 1 be fulfilled. Let, moreover,* (18) *and* (19) *hold. Then inclusion* (13) *holds if and only if*

$$\lim_{t\to t_0}\bigg\|\Phi^{-1}(t)\int\limits_{t_0}^{t}\big(X_0(s)X_m^{-1}(s)q_m(s)-q(s)\big)\,ds\bigg\|=0\ \ (m=1,2,\dots),$$

and

$$\lim_{m\to+\infty}\|\Phi^{-1}(t)(X_m(t)-X_0(t))\|=0,$$

$$\lim_{m\to+\infty}\bigg\|\Phi^{-1}(t)\bigg|\int\limits_{t_0}^{t}\big(X_0(s)X_m^{-1}(s)q_m(s)-q(s)\big)\,ds\bigg| \\ +\bigg|\int\limits_{t_0}^{t}\Phi^{-1}(s)P_*(s)\bigg|\int\limits_{t_0}^{s}\big(X_0(\tau)X_m^{-1}(\tau)q_m(\tau)-q(\tau)\big)\,d\tau\bigg|\,ds\bigg|\bigg\|=0,$$

hold uniformly on I_{t_0}, where X_0 and X_m $(m=1,2,\dots)$ are the fundamental matrices of systems (1) *and* (3)*, respectively.*

Corollary 1. *Let conditions of Theorem 1 be fulfilled for $q(t)\equiv 0_n$. Let, moreover,* (18) *and* (19) *hold. Then inclusion* (13) *holds if and only if*

$$((P_m,0_n))_{m=1}^{+\infty}\in\mathcal{S}_{P_*}(P,0_n;\Phi,t_0). \tag{20}$$

Remark 2. In Theorem 4′, as in Corollary 1, condition (18) is essential and it cannot be neglected, i.e., if the condition is violated, then the conclusion of the theorem and the corollary is not true. We present an example.

Let $I=[0,1]$, $n=1$, $t_0=0$, $B=0$, $B_0=1$, $\Phi(t)\equiv t$; $P(t)=P_m(t)=P_*(t)\equiv t^{-1}$ $(m=1,2,\dots)$, $q(t)\equiv 0$, $q_m(t)\equiv m^{-1}\cos(m^{-1}\ln t)$ $(m=1,2,\dots)$. Then

$$C_*(t,\tau)\equiv t\tau^{-1},\quad x_0(t)\equiv 0,\quad x_m(t)\equiv t\sin\frac{\ln t}{m}\ \ (m=1,2,\dots).$$

So, all the conditions of Theorem 4′ are fulfilled, except of (18), but condition (8) is not fulfilled uniformly on I. On other hand, this means that condition (20) is fulfilled but condition (13) is violated.

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