On Initial-Boundary Value Problems for Quasilinear Hyperbolic Systems of Second Order

Maram M. Alrumayh

Florida Institute of Technology, Melbourne, USA

E-mail: malsalem2017@my.fit.edu

In the rectangle $\Omega=[0,a]\times[0,b]$ consider the nonlinear hyperbolic system

$$u_{xy}=f(x,y,u_x,u_y,u), \tag{1}$$

$$u(0,y)=\varphi(y), \quad h(u_x(x,\cdot\,))(x)=\psi'(x), \tag{2}$$

where $f:\Omega\times\mathbb{R}^{3n}\to\mathbb{R}^n$ is a continuous vector function that is continuously differentiable with respect to the first $2n$ phase variables, $\varphi\in C^1([0,b];\mathbb{R}^n)$, $\psi\in C^1([0,a];\mathbb{R}^n)$, and $h:C([0,b];\mathbb{R}^n)\to C([0,a];\mathbb{R}^n)$ is a bounded linear operator.

Let $v=(v_1,\dots,v_n)$, $w=(w_1,\dots,w_n)$ and $z=(z_1,\dots,z_n)$. For a function $f(x,y,v,w,u)$ that is continuously differentiable with respect to v, w and u, set:

$$F_1(x,y,v,w,z)=\frac{\partial f(x,y,v,w,z)}{\partial v}\,, \quad F_2(x,y,v,w,z)=\frac{\partial f(x,y,v,w,z)}{\partial w}\,,$$

$$F_0(x,y,v,w,z)=\frac{\partial f(x,y,v,w,z)}{\partial z}\,,$$

$$P_j[u](x,y)=F_j\big(x,y,u_x(x,y),u_y(x,y),u(x,y)\big) \ \ (j=0,1,2).$$

$C^{1,1}(\Omega;\mathbb{R}^n)$ is the Banach space of continuous vector functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives u_x, u_y, u_{xy}, endowed with the norm

$$\|u\|_{C^{1,1}}=\|u\|_C+\|u_x\|_C+\|u_y\|_C+\|u_{xy}\|_C.$$

$C^1(\Omega;\mathbb{R}^n)$ is the Banach space of continuous vector functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives u_x, u_y, endowed with the norm

$$\|u\|_{C^{1,1}}=\|u\|_C+\|u_x\|_C+\|u_y\|_C.$$

If $u_0\in C(\Omega:\mathbb{R}^n)$ and $r>0$, then

$$\mathbf{B}(u_0;r)=\big\{u\in C(\Omega:\mathbb{R}^n): \ \ \|u-u_0\|\le r\big\}.$$

If $u_0\in C^1(\Omega:\mathbb{R}^n)$ and $r>0$, then

$$\mathbf{B}^1(u_0;r)=\big\{u\in C^1(\Omega:\mathbb{R}^n): \ \ \|u-u_0\|_{C^1}\le r\big\}.$$

Definition 1. Let u_0 be a solution of problem (1), (2), and $r>0$. Problem (1), (2) is said to be (u_0,r)-*well-posed* if:

(i) $u_0(x,y)$ is the unique solution of the problem in the ball $\widetilde{\mathcal{B}}^1(u_0;r)$;

(ii) There exists $\varepsilon_0 > 0$ such that for an arbitrary $\varepsilon > 0$ and $M > 0$ there exists $\delta > 0$ such that for any $\widetilde{f}(x,y,v,w,z)$ that is continuously differentiable with respect to v and w, $\widetilde{\varphi} \in C^1([0,b];\mathbb{R}^n)$, $\widetilde{\psi} \in C^1([0,a];\mathbb{R}^n)$, satisfying the inequalities

$$\begin{aligned} \Big\|\frac{\partial \widetilde{f}(x,y,v,w,z)}{\partial v}\Big\| &\le \varepsilon_0 \ \text{ for } \ (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n} \\ \Big\|\frac{\partial \widetilde{f}(x,y,v,w,z)}{\partial w}\Big\| &\le M \ \text{ for } \ (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}, \end{aligned} \tag{3}$$

$$\|\widetilde{f}(x,y,v,w,z)\| \le \delta \ \text{ for } \ (x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}, \quad \|\widetilde{\varphi}\|_{C^1([0,b])} + \|\widetilde{\psi}\|_{C^1([0,a])} \le \delta, \tag{4}$$

the problem

$$u_{xy} = f(x,y,u_x,u_y,u) + \widetilde{f}(x,y,u_x,u_y,u), \tag{$\widetilde{1}$}$$

$$u(0,y) = \varphi(y) + \widetilde{\varphi}(y), \quad h\big(u_x(x,\cdot)\big)(x) = \psi'(x) + \widetilde{\psi}'(x), \tag{$\widetilde{2}$}$$

has at least one solution in the ball $\mathbf{B}^1(u_0;r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0;\varepsilon)$.

Definition 2. Let u_0 be a solution of problem (1), (2), and $r > 0$. Problem (1), (2) is said to be *strongly* (u_0,r)-*well-posed* if:

(i) Problem (1), (2) is (u_0,r)-well-posed;

(ii) There exist positive numbers M_0 and δ_0 such that for arbitrary $\delta \in (0,\delta_0)$, $\widetilde{f}(x,y,v,w,z)$ that is continuously differentiable with respect to v and w, $\widetilde{\varphi} \in C^1([0,b];\mathbb{R}^n)$ and $\widetilde{\psi} \in C^1([0,a];\mathbb{R}^n)$, satisfying the inequalities (3) and (4), problem $(\widetilde{1})$, $(\widetilde{2})$ has at least one solution in the ball $\mathbf{B}^1(u_0;r)$, and each such solution belongs to the ball $\mathbf{B}^1(u_0;M_0\,\delta)$.

Definition 3. Problem (1), (2) is called *well-posed* (*strongly well-posed*) if it has a unique solution u_0 and it is (u_0,r)-well-posed (strongly (u_0,r)-well-posed) for every $r > 0$.

Consider the boundary value problem for the system of nonlinear ordinary differential equations

$$z' = p(t,z), \quad \ell(z) = c, \tag{5}$$

where $p \in C([0,b] \times \mathbb{R}^n;\mathbb{R}^n)$, $c \in \mathbb{R}^n$ and $\ell : C([0,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a bounded linear operator.

Definition 4. Let z_0 be a solution of problem (5), and $r > 0$. Problem (5) is said to be (z_0,r)-*well-posed* if:

(i) $z_0(t)$ is the unique solution of the problem in the ball $\mathbf{B}(z_0;r)$;

(ii) For an arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that for any $\widetilde{c}$, and $\widetilde{p} \in C([0,b] \times \mathbb{R}^n)$ satisfying the inequalities

$$\big\|c - \widetilde{c}\big\| < \delta, \quad \|p - \widetilde{p}\|_C < \delta \tag{6}$$

the problem

$$z' = \widetilde{p}(t,z), \quad \ell(z) = \widetilde{c}, \tag{$\widetilde{5}$}$$

has at least one solution in the ball $\mathbf{B}(z_0;r)$, and each such solution belongs to the ball $\mathbf{B}(z_0;\varepsilon)$.

Definition 4 is a slight modification of Definition 3.2 from [1]. Definition 1 is an adaptation of the idea of Definition 4 to problem (1), (2).

Definition 5. Let u_0 be a solution of problem (5), and $r > 0$. Problem (5) is said to be *strongly* (z_0, r)*-well-posed* if:

(i) $z_0(t)$ is the unique solution of the problem in the ball $\mathbf{B}(z_0; r)$;

(ii) There exist positive numbers M and δ_0 such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{c}_k$, and $\widetilde{p} \in C([0,b] \times \mathbb{R}^n)$ satisfying inequalities (6), problem $(\widetilde{5})$ has at least one solution in the ball $\mathbf{B}(z_0; r)$, and each such solution belongs to the ball $\mathbf{B}(z_0; M\,\delta)$.

Remark 1. It is obvious that strong well-posedness implies well-posedness. The converse, however, is not true. As an example, consider the problem

$$z' = z^3, \quad z(0) = z(\omega), \tag{7}$$

which is well-posed and has the unique solution $z_0(t) \equiv 0$. The perturbed problem

$$z' = z^3 - \delta, \quad z(0) = z(b)$$

has the unique solution $z_\delta(t) = \delta^{\frac{1}{3}}$. It is clear that there exists no positive number M such that $\delta^{\frac{1}{3}} \leq M\delta$ as $\delta \to 0$. Consequently, problem (7) is not strongly well-posed.

Definition 6. A solution z_0 of problem (5) is said to be strongly isolated, if problem (5) is strongly (z_0, r)-well-posed for some $r > 0$.

Remark 2. The concept of a strongly isolated solution of a nonlinear boundary value problem was introduced in [1]. However, our definition of a strongly isolated solution is a modification of Definition 3.1 from [1]. Also, Corollary 3.6 from [1] implies that if the vector function $p(t,z)$ is continuously differentiable with respect to the phase variables, then strong isolation of a solution z_0 is equivalent to the fact that the linear homogeneous problem

$$z' = P(t)z, \quad \ell(z) = 0, \tag{8}$$

has only the trivial solution, where $P(t) = \frac{\partial p}{\partial z}(t, z_0(t))$.

Theorem 1. *Let f be a continuously differentiable function with respect to the phase variables v, w and z, and let u_0 be a solution of problem $(1), (2)$. Then, problem $(1), (2)$ is strongly (u_0, r)-well-posed for some $r > 0$, if and only if the linear homogeneous problem*

$$u_{xy} = P_0(x,y)u + P_1(x,y)u_x + P_2(x,y)u_y, \tag{1_0}$$
$$u(0,y) = 0, \quad h(u_x(x,\cdot))(x) = 0, \tag{2_0}$$

where $P_j(x,y) = P_j[u_0](x,y)$ $(j = 0,1,2)$, is well-posed.

Theorem 2. *Problem $(1_0), (2_0)$ is well-posed if and only if the linear homogeneous problem*

$$\frac{dz}{dy} = P_1(x,y)z, \quad h(z)(x) = 0$$

has only the trivial solution for every $x \in [0,a]$.

Remark 3. The sufficiency part of Theorem 2 was proved in [2] (see Theorems 4.1 and 4.1′). Similar theorem for higher order linear hyperbolic equations for proved in [4] (see Theorem 1.1).

Theorem 3. *Let f be a continuously differentiable with respect to the phase variables v, w and z, and let there exist matrix functions $Q_i \in C(\Omega; \mathbb{R}^{n\times n})$ $(i = 1,2)$ and a positive constant ρ such that:*

(A_1) $\|F_0(x,y,v,w,z)\| + \|F_2(x,y,v,w,z)\| \le \rho$ *for* $(x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}$;

(A_2) $Q_1(x,y) \le F_1(x,y,v,w,z) \le Q_2(x,y)$ *for* $(x,y,v,w,z) \in \Omega \times \mathbb{R}^{3n}$;

(A_3) *for every* $x \in [0,a]$ *and arbitrary measurable matrix function* $P : [0,b] \to \mathbb{R}^{n\times n}$ *satisfying the inequalities*

$$Q_1(x,y) \le P(y) \le Q_2(x,y) \ \ \textit{for} \ \ y \in [0,b],$$

problem (8) *has only the trivial solution. Then problem* (1), (2) *is strongly well-posed.*

Theorem 4. *Let* f *be a continuously differentiable function with respect to the phase variables* v, w *and* z, *and let* v_0 *be a strongly isolated solution of the problem*

$$v' = p(y,v), \quad h(v)(0) = \psi'(0), \tag{9}$$

where

$$p(y,v) = f\big(0,y,v,\varphi'(y),\varphi(y)\big).$$

Then there exists $\alpha \in (0,a]$ *such that in the rectangle* $\Omega_\alpha = [0,\alpha] \times [0,b]$ *problem* (1), (2) *has a unique solution* u *satisfying the condition*

$$u_x(0,y) = v_0(y) \ \ \textit{for} \ \ y \in [0,b].$$

Remark 4. Conditions of Theorem 4 do not guarantee unique solvability of problem (1), (2). Indeed, consider the problem

$$u_{xy} = \prod_{k=1}^{m} \big(u_x - k\big) + x\, f_0(x,y,u_x,u_y,u), \tag{10}$$

$$u(0,y) = 0, \ \ u^{(1,0)}(x,0) = u^{(1,0)}(x,b), \tag{11}$$

where $f_0 : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ is a continuously differentiable function. For this case problem (9) has the form

$$v' = \prod_{k=1}^{m} (v-k), \quad v(0) = v(b).$$

The latter problem has exactly m strongly isolated solutions $v_k = k\pi$ $(k = 1,\dots,m)$. By Theorem 4, for every integer $k \in \{1,\dots,m\}$ there exists $\alpha_k > 0$ such that in $\Omega_{\alpha_k} = [0,\alpha_k] \times [0,b]$, problem (10), (11) has a unique solution u_k satisfying the condition

$$u_k^{(1,0)}(0,y) = k \ \ \text{for} \ \ y \in [0,b].$$

Consider the family of problems

$$z' = p_\lambda(t,z), \quad \ell_\lambda(z) = c_\lambda, \tag{12_λ}$$

where $\lambda \in \Lambda$, $p_\lambda \in C([0,b] \times \mathbb{R}^n; \mathbb{R}^n)$, $\ell_\lambda : C([0,b]) \to \mathbb{R}^n$ are bounded linear functionals, and $c_\lambda \in \mathbb{R}^n$.

Let for $\lambda \in \Lambda$ and $r > 0$, z_λ be a solution of problem (12_λ). The family of problems (12_λ) $(\lambda \in \Lambda)$ is said to be *uniformly strongly* (z_λ, r)*-well-posed*, if:

(i) z_λ is unique in the ball $\mathbf{B}(z_\lambda; r)$;

(ii) There exist positive numbers M and δ_0 *independent of* λ such that for arbitrary $\delta \in (0, \delta_0)$, $\widetilde{c} \in \mathbb{R}^n$, and $\widetilde{p}_\lambda \in C([0,b] \times \mathbb{R}^n; \mathbb{R}^n)$ satisfying the inequalities

$$\|c - \widetilde{c}\| < \delta, \quad \|p_\lambda - \widetilde{p}_\lambda\|_C < \delta,$$

the problem

$$z' = \widetilde{p}_\lambda(t, z), \quad \ell_\lambda(z) = \widetilde{c}_\lambda, \tag{$\widetilde{12}\lambda$}$$

has at least one solution in the ball $\mathbf{B}(z_\lambda; r)$, and each such solution belongs to the ball $\mathbf{B}(z_\lambda; M\delta)$.

A family of solutions $\{z_\lambda\}_{\lambda \in \Lambda}$ is said to be *uniformly strongly isolated* if the family of problems (12_λ) $(\lambda \in \Lambda)$ is *uniformly strongly* (z_λ, r)*-well-posed* for some $r > 0$.

Let $J = [0, \alpha)$, $\alpha \in (0, a]$, $(J = [0, \alpha]$, $\alpha \in (0, a))$, and u be a solution of problem (1), (2) in the rectangle $J \times [0, b]$. u is called *continuable*, if there exists $\alpha_1 \in [\alpha, a]$ $(\alpha_1 \in (\alpha, a])$ and a solution u_1 of problem (1), (2) in $[0, \alpha_1] \times [0, b]$ such that

$$u_1(x, y) = u(x, y) \ \text{ for } \ (x, y) \in [0, \alpha) \times [0, b].$$

Otherwise u is called *non-continuable*.

Theorem 5. *Let u be a a non-continuable solution of problem* (1), (2) *defined on $J \times [0, b]$, and let for every $x_0 \in J$, $v(y) = u_x(x_0, y)$ be a solution of the problem*

$$v' = p[u](x_0, y, v), \quad h(v)(x_0) = \psi(x_0). \tag{13}$$

If the family of solutions $v(y) = u_x(x_0, y)$ $(x_0 \in J)$ is uniformly strongly isolated, then either $J = [0, a]$, or $J = [0, \alpha)$ and

$$\lim_{x \to \alpha} \Big(\|u_x(x, \cdot\,)\|_{C([0,b])} + \|u(x, \cdot\,)\|_{C([0,b])} + \|u_y(x, \cdot\,)\|_{C([0,b])} \Big) = +\infty. \tag{14}$$

Definition 7. Let u be a non-continuable solution of problem (1), (2) in $J \times [0, b]$ and let $\alpha = \sup J$. We say that a measurable matrix function $P : [0, b] \to \mathbb{R}^{n \times n}$ belongs to the set $S_f^\alpha[u]$, if there exists an increasing sequence $x_k \uparrow \alpha$ as $k \to \infty$ such that

$$\lim_{k \to \infty} \int_0^y P_1[u](x_k, t)\, dt = \int_0^y P(t)\, dt$$

uniformly on $[0, b]$.

Corollary. *Let u be a non-continuable solution of problem* (1), (2) *in $J \times [0, b]$, and let $\alpha = \sup J$. If for an arbitrary $P \in S_f^\alpha[u]$ the homogeneous problem*

$$z' = P(t) z, \quad h(z)(\alpha) = 0$$

has only the trivial solution, then either $J = [0, a]$, or $J = [0, \alpha)$ and (14) *holds.*

References

[1] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Translated in *J. Soviet Math.* **43** (1988), no. 2, 2259–2339. *Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.

[2] T. Kiguradze, Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Mem. Differential Equations Math. Phys.* **1** (1994), 1–144.

[3] T. Kiguradze and R. Ben-Rabha, On strong well-posedness of initial-boundary value problems for higher order nonlinear hyperbolic equations with two independent variables. *Georgian Math. J.* **24** (2017), no. 3, 409–428.

[4] T. I. Kiguradze and T. Kusano, On the well-posedness of initial-boundary value problems for higher-order linear hyperbolic equations with two independent variables. (Russian) *Differ. Uravn.* **39** (2003), no. 4, 516–526; translation in *Differ. Equ.* **39** (2003), no. 4, 553–563.