Disconjugacy and Green’s Functions Sign for Some Two-Point Boundary Value Problems for Fourth Order Ordinary Differential Equations

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1 Introduction

Here we consider the question of the disconjugacy on the interval $I := [a, b] \subset [0, +\infty[$ of the fourth order linear ordinary differential equation

$$
(4)(t) = p(t)u(t) - \mu u(t) \quad \text{for} \quad t \in I,
$$

when $p : I \to \mathbb{R}$ is Lebesgue integrable function, $\mu \in \mathbb{R}$, and the question of the Green’s functions sign for equation (1.1) under one of the following two-point boundary conditions

$$
\begin{align*}
&u(a) = 0, \quad u^{(i)}(b) = 0 \quad (i = 0, 1, 2), \\
&u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \quad (i = 0, 1), \\
&u^{(i)}(a) = 0 \quad (i = 0, 1, 2), \quad u(b) = 0.
\end{align*}
$$

There are established the optimal sufficient conditions of disconjugacy of equation (1.1) when the coefficient $p$ is not necessarily constant sign function. On the basis of these results we prove the necessary and sufficient conditions of non-negativity (non-positivity) of Green’s function for problems (1.1), (1.21) ($\ell \in \{1, 2, 3\}$), which are formulated in a terminology of eigenvalues of problems under the consideration.

Here we use the following notations.

$\mathbb{R} = ]-\infty, +\infty[,$ $\mathbb{R}_0 = ]-\infty, 0[,\mathbb{R}_0^+ = [0, +\infty[.$

$C(I; \mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$C^3(I; \mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)| \, ds$.

For arbitrary $x, y \in L(I; \mathbb{R})$, the notation

$$
x(t) \preceq y(t) \quad (x(t) \succ y(t)) \quad \text{for} \quad t \in I,
$$

means that $x \leq y$ ($x \geq y$) and $x \neq y$.

Also we use the notation $[x]_\pm = \frac{|x| \pm x}{2}$.

By a solution of equation (1.1) we understand a function $u \in C^3(I; \mathbb{R})$ which satisfies equation (1.1) a.e. on $I$.

Also we need the following definition.
Definition 1.1. Equation
\[ u^{(4)}(t) = p(t)u(t) \] \hspace{1cm} (1.3)
is said to be disconjugate (non-oscillatory) on \( I \), if every nontrivial solution \( u \) has less then four zeros on \( I \), the multiple zeros being counted according to their multiplicity. Otherwise we say that equation (1.3) is oscillatory on \( I \).

The given study is based on our previous results from the paper [3] and the results of A. Cabada and R. Enguica from [1]. For the formulation of our main results we need the following definitions and propositions from our previous paper.

Definition 1.2. We will say that \( p \in D_+(I) \) if \( p \in L(I; \mathbb{R}_0^+) \), and problem (1.3), (1.2) has a solution \( u \) such that
\[ u(t) > 0 \quad \text{for } t \in [a, b[. \]

Definition 1.3. We will say that \( p \in D_-(I) \) if \( p \in L(I; \mathbb{R}_0^-) \), and problem (1.3), (1.2) has a solution \( u \) such that
\[ u(t) > 0 \quad \text{for } t \in [a, b[. \]

The propositions below are Theorems 2, 4, and 6, respectively, from the paper [3].

Proposition 1.1. Let \( p \in L(I; \mathbb{R}_0^+) \). Then equation (1.3) is disconjugate on \( I \) iff there exists \( p^* \in D_+(I) \) such that
\[ p(t) \preceq p^*(t) \quad \text{on } I. \]

Proposition 1.2. Let \( p \in L(I; \mathbb{R}_0^-) \). Then equation (1.3) is disconjugate on \( I \) iff there exists \( p_* \in D_-(I) \) such that
\[ p(t) \succeq p_*(t) \quad \text{on } I. \]

Proposition 1.3. Let \( p_* \in D_-(I) \) and \( p^* \in D_+(I) \). Then for an arbitrary function \( p \in L(I; \mathbb{R}) \) such that
\[ p_*(t) \preceq -\lfloor p(t) \rfloor_-, \quad [p(t)]_+ \preceq p^*(t) \quad \text{on } I, \] \hspace{1cm} (1.4)
equation (1.3) is disconjugate on \( I \).

Remark 1.1. From Proposition 1.1 (Proposition 1.2) it is clear that the structure of the set \( D_+(I) \) (\( D_-(I) \)) is such that if \( x, y \in D_+(I) \) \((x, y \in D_-(I))\), then none of the inequalities \( x \preceq y \) and \( y \preceq x \) hold.

Remark 1.2. If \( \lambda_1 \) \((\lambda_2)\) is the first positive eigenvalue of the problem
\[
\begin{align*}
\quad \quad u^{(4)}(t) &= \lambda^4 u(t), \quad u^{(i)}(0) = 0, \quad u^{(i)}(1) = 0 \quad (i = 0, 1) \\
(u^{(4)}(t) &= -\lambda^4 u(t), \quad u^{(i)}(0) = 0 \quad (i = 0, 1, 2), \quad u(1) = 0),
\end{align*}
\]
then
\[ \frac{\lambda_1^4}{(b-a)^4} \in D_+(I) \quad \left( \frac{\lambda_2^2}{(b-a)^2} \in D_-(I) \right). \]

Also it is well-known (see [1] or [2]), that \( \lambda_1 \approx 4.73004 \) and \( \lambda_2 \approx 5.553 \).
2 Main results

First we consider the results on disconjugacy of equation (1.1) on the interval \([a, b]\).

**Theorem 2.1.** Let \(p \in L(I; \mathbb{R})\), conditions

\[
\alpha_p := \inf_{p^* \in D_+(I)} \left\{ \text{ess sup}_{t \in I} \{ p(t) - p^*(t) \} \right\} \leq 0, \\
\beta_p := \sup_{p_* \in D_-(I)} \left\{ \text{ess inf}_{t \in I} \{ p(t) - p_*(t) \} \right\} \geq 0, \\
\tag{2.1}
\]

hold, and

\(\alpha_p \neq \beta_p\).

Then equation (1.1) is disconjugate on \(I\) if \(\mu \in [\alpha_p, \beta_p]\).

**Remark 2.1.** Theorem 2.1 is optimal in the sense that there exists \(p \in L(I; \mathbb{R})\) such that if \(\mu = \alpha_p\) or \(\mu = \beta_p\), then equation (1.1) is oscillatory on \(I\). Indeed, let \(p(t) \equiv \frac{\lambda_1^4 - \lambda_2^4}{2(b-a)^4}\), where due to Remark 1.2 we have \(\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)\) and \(-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)\). Then from Remark 1.1 it immediately follows that

\[
\alpha_p = p(t) - \frac{\lambda_1^4}{(b-a)^4} \quad \text{and} \quad \beta_p = p(t) + \frac{\lambda_2^4}{(b-a)^4}.
\]

Therefore if \(\mu = \alpha_p\) or \(\mu = \beta_p\), then equation (1.1) is oscillatory on \(I\).

**Corollary 2.1.** Let \(p \in D_+(I)\). Then \(\beta_p > \frac{\lambda_1^4}{(b-a)^4}\), and equation (1.1) is disconjugate on \(I\) if \(\mu \in [0, \beta_p]\).

**Corollary 2.2.** Let \(p \in D_-(I)\). Then \(\alpha_p < -\frac{\lambda_2^4}{(b-a)^4}\), and equation (1.1) is disconjugate on \(I\) if \(\mu \in [\alpha_p, 0]\).

From the last two corollaries we immediately have

**Corollary 2.3.** Let \(\mu \in [0, \frac{\lambda_1^4}{(b-a)^4}] \) \((\mu \in [-\frac{\lambda_2^4}{(b-a)^4}, 0])\). Then equation (1.1) is disconjugate on \(I\) for an arbitrary \(p \in D_+(I)\) \((p \in D_-(I))\).

**Remark 2.2.** Corollaries 2.1 and 2.2 are optimal.

As it is well-known the disconjugacy is only a sufficient condition in order to ensure the constant sign of Green’s function of problems (1.3), (1.2\(\ell\)) \((\ell \in \{1, 2, 3\})\). For this reason we introduce here theorems with necessary and sufficient conditions which guarantee that Green’s function of problems (1.1), (1.2\(\ell\)) or (1.1), (1.2\(3\)) will be the constant sign function. Also we will find such coefficients \(p\), and such values of the parameter \(\mu\), for which Green’s functions of the mentioned problems are constant sign functions but equation (1.1) is oscillatory on \(I\) (see Remark 2.3).

**Theorem 2.2.** Let \(p \in D_+(I) \cap C(I; \mathbb{R})\). Then:

(a) Green’s function of problem (1.1), (1.2\(\ell\)) is non-negative on \(I \times I\) if \(\mu \in [0, \mu_p]\), where \(\mu_p := \min\{\mu^*_1, \mu^*_3\}\), \(\mu^*_\ell\) \((\ell = 1, 3)\) is the first positive eigenvalue of problem (1.1), (1.2\(\ell\));

(b) The estimation \(\mu_p \geq \beta_p > \frac{\lambda_1^4}{(b-a)^4}\) is valid.

**Theorem 2.3.** Let \(p \in D_-(I) \cap C(I; \mathbb{R})\). Then:
(a) Green’s function of problem (1.1), (1.2) is non-negative on $I \times I$ iff $\mu \in [\mu_p, 0]$, where $\mu_p$ is the biggest negative eigenvalue of problem (1.1), (1.2);

(b) The estimation $\mu_p \leq \alpha_p \leq -\frac{\lambda_1^4}{(b-a)^4}$ is valid.

**Theorem 2.4.** Let $p \in D_+ (I) \cap C(I; \mathbb{R})$. Then:

(a) Green’s function of problem (1.1), (1.2) is non-positive on $I \times I$ iff $\mu \in [0, \mu_p]$, where $\mu_p$ is the first positive eigenvalue of problem (1.1), (1.2);

(b) The estimation $\mu_p \leq \beta_p > \frac{\lambda_2^4}{(b-a)^4}$ is valid.

**Theorem 2.5.** Let $p \in D_- (I) \cap C(I; \mathbb{R})$. Then:

(a) Green’s function of problem (1.1), (1.2) is non-positive on $I \times I$ iff $\mu \in [\mu_p, 0]$, where $\mu_p$ is the biggest negative eigenvalue of problem (1.1), (1.2);

(b) The estimation $\mu_p \leq \alpha_p < -\frac{\lambda_1^4}{(b-a)^4}$ is valid.

**Remark 2.3.** In Theorems 2.2 and 2.4 (Theorems 2.3 and 2.5) from the definition of the number $\mu_p$ it is clear that equation (1.1) is oscillatory on $I$ if $\mu = \mu_p$. Therefore from Corollary 2.1 (Corollary 2.2) it immediately follows that

$$
\mu_p \geq \beta_p > \frac{\lambda_2^4}{(b-a)^4} \quad \left( \mu_p \leq \alpha_p < -\frac{\lambda_1^4}{(b-a)^4} \right).
$$

**References**

