An Approach to Two-Sided Pointwise Estimating Solutions of Linear Boundary Value Problems with an Uncertainty

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1 Introduction

In this communication we are concerned with the question on estimates of solutions to the general linear boundary value problem

$$\mathcal{L}x = f, \quad \ell x = 0, \tag{1.1}$$

in the case of an uncertainty with respect to the right-hand side of the functional differential system $\mathcal{L}x = f$. The linear operator $\mathcal{L} : AC^n \to L^n$ is assumed to be bounded and to have a Fredholm principal part [1, pp. 7, 42]; $\ell : AC^n \to R^n$ is linear bounded vector-functional with linearly independent components, $f \in L^n$. Here L^n is the space of summable functions $f : [0, T] \to R^n$, AC^n is the space of absolutely continuous functions $x : [0, T] \to R^n$. The spaces L^n , AC^n are assumed to be equipped with natural norms.

The right-hand side f is assumed to be known with an uncertainty, namely, it is given only that its values are constrained by the inequalities

$$\Lambda \cdot f(t) \leq \gamma, \quad t \in [0, T], \tag{1.2}$$

with a constant $(N \times n)$-matrix Λ and $\gamma \in R^N$. We assume that the solutions set, V, to the system $\Lambda v \leq \gamma$ is nonempty and bounded.

The question we discuss here is one of the two-sided estimates of any solution to (1.1) at a fixed point, say, $\tau \in [0, T]$:

$$q^1 \leq x(\tau) \leq q^2. \tag{1.3}$$

The consideration is based on the Green operator

$$G : L^n \to AC^n, \quad (Gf)(t) = \int\limits_0^T G(t, s) f(s)\, ds \tag{1.4}$$

of (1.1). For the existence of G and the integral representation of it we refer to [1, pp. 46–49]. Recall that the matrix kernel $G(t, s)$ is called the Green matrix to (1.1). We can understand the estimate (1.3) as an external estimate of the range to G over all f's such that $f(t) \in V$ for almost all $t \in [0, T]$.

2 The two-sided estimation of solutions

We will use the following notation.

Fix a τ from the segment $[0,T]$. Let $(e_1,\dots,e_n)$ be the canonical basis in R^n, thus e_i has 1 as the i-th component and zero as the rest ones. Introduce the vector $e_i^k=(-1)^{k-1}e_i$, $i=1,\dots,n$, $k=1,2$. Define $G_i^k(s)=(e_i^k)'\cdot G(\tau,s)$, where $(\,\cdot\,)'$ stands for transposition.

Denote by $w_i^k(s)$ the solution of the problem $G_i^k(s)\cdot v\to\max$, $v\in V$. Fix a collection of s_j, $j=0,\dots,\mu$, $0=s_0<s_1<\dots<s_\mu=T$, and define $\widetilde{w}_i^k(s)=\sum\limits_{j=1}^{\mu}\chi_{[s_{j-1},s_j)}(s)w_i^k(s_j)$, where $\chi_A(s)$ is the characteristic function of a set $A\subset R$.

Theorem. *Let $G(\tau,s)$ be piecewise continuous in s on $[0,T]$ and nonnegative δ_i^k, $i=1,2,\dots,n$, $k=1,2$ be such that the inequalities*

$$\int\limits_0^T G_i^k(\tau,s)w_i^k(s)\,ds\le\int\limits_0^T G_i^k(\tau,s)\widetilde{w}_i^k(s)\,ds+\delta_i^k=q_i^k,\ \ i=1,2,\dots,n,\ k=1,2,$$

hold. Then, for any f constrained by (1.2)*, there take place the estimates of $x(\tau)$:*

$$(e_i^k)'x(\tau)\le q_i^k,\ \ i=1,2,\dots,n,\ k=1,2.$$

Example. Let us consider the system (see [3])

$$\begin{aligned}\dot{x}_1(t)&=x_2(t-1)+f_1(t),\\ \dot{x}_2(t)&=-x_2(t)+f_2(t),\end{aligned}\quad t\in[0,3], \tag{2.1}$$

where $x_2(s)=0$ if $s<0$. Set up the boundary conditions by the equality

$$\ell x\equiv x(3)-x(0)=0. \tag{2.2}$$

As for the right-hand side f, the information about it is confined only to the inequalities

$$\begin{gathered}-0.25\le f_1(t)\le-0.15,\quad 0.1\le f_2(t)\le0.5,\\ 0.4f_1(t)-0.1f_2(t)\ge-0.11,\quad 0.4f_1(t)+0.1f_2(t)\le-0.05.\end{gathered} \tag{2.3}$$

The boundary value problem (2.1), (2.2) is iniquely solvable since (2.1) has the fundamental matrix

$$X(t)=\begin{pmatrix}1 & \chi_{[1,3]}(t)(1-e^{1-t})\\ 0 & e^{-t}\end{pmatrix},$$

and

$$\ell X=\begin{pmatrix}-1 & 1-e^{-2}\\ 0 & e^{-3}-2\end{pmatrix},$$

with $\det\ell X=2-e^{-3}$.

Put $\tau=2$ and obtain the estimate of $x(2)$ that holds for any f constrained by (2.3).

Having in mind the Cauchy matrix to the system (2.1) constructed in [3], we can construct the Green matrix $G(t,s)$ to (2.1), (2.2). For purpose of estimating $x(2)$, it suffices to use the section

$G(2,s)$. Let us give its description component by component:

$$G_{11}(2,s)=\begin{cases}2, \ s\in[0,2],\\ 1, \ s\in(2,3];\end{cases}$$

$$G_{12}(2,s)=\begin{cases}-\dfrac{e^{-3}-2+2e^{s-2}-2e^{s-3}+e^{s-4}}{2-e^{-3}}+1-e^{s-1}, & s\in[0,2],\\[2mm] -\dfrac{e^{-3}-2+2e^{s-2}-2e^{s-3}+e^{s-4}}{2-e^{-3}}\,, & s\in(2,3];\end{cases}$$

$$G_{22}(2,s)=0;\quad G_{11}(2,s)=\begin{cases}\dfrac{e^{s-5}}{2-e^{-3}}+e^{s-2}, & s\in[0,2],\\[2mm] \dfrac{e^{s-5}}{2-e^{-3}}\,, & s\in(2,3].\end{cases}$$

In this case Theorem gives the following estimates:

$$-1.19\le x_1(2)\le -0.52;\quad 0.1\le x_2(2)\le 0.28.$$

To illustrate the interrelation between the rigidity of constraints and the size of the values set to $x(2)$, we note that in the case of $-0.01\le f_1(t)\le 0.01$, $-0.01\le f_2(t)\le 0.01$, we obtain $-0.07\le x_1(2)\le 0.07$, $-0.01\le x_2(2)\le 0.01$. Clear, it depends on the Green operator property, but the approach we discuss opens a way to take into account specific properties of solution components in contrary to the estimates in terms of the norms introduced into the corresponding functional spaces.

In conclusion we refer to the papers [2–7] where different aspects of the problem on enclosing solutions to various classes of dynamic systems are presented and useful references can be found.

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