# Application of the Averaging Method to Optimal Control Problems of Systems with Impulse Action in Non-Fixed Moments of Times

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We study the application of the method of averaging to the problems of optimal control over impulsive differential equations. The procedure of averaging allows to replace the original problem with the problem of optimal control by a system of ordinary differential equations. The optimal control problems are investigated on finite and infinite horizons.

## Introduction

For a system of differential equations with an impulsed action at non-fixed moments of time

$$\begin{gathered} \dot{x} = \varepsilon X(t,x,u), \ \ t \neq t_i(x), \\ \triangle x\big|_{t=t_i(x)} = \varepsilon I_i(x,v_i), \\ x(0) = x_0, \ \ t_i(x) < t_{i+1}(x), \end{gathered} \tag{0.1}$$

two optimal control problems on a finite and infinite interval with a quality criterion are considered:

(1) on a finite interval with a quality criterion are considered:

$$J^1_\varepsilon(u,v) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} \Phi(t,x(t),u(t))\,dt + \varepsilon \sum_{0<t_i(x)<\frac{T}{\varepsilon}} \Psi_i(x(t_i),v_i) \longrightarrow \inf, \tag{0.2}$$

(2) on an infinite interval with a quality criterion are considered:

$$J^2_\varepsilon(u,v) = \varepsilon \int\limits_0^{\infty} e^{-\gamma t} L(t,x(t))\,dt \longrightarrow \inf. \tag{0.3}$$

Here $T>0$, $\varepsilon>0$, $\gamma>0$ are fixed; $t\geq 0$, $x\in D$ is a domain in the space $R^d$, $u\in U\subset R^m$, $v_i\in V\subset R^r$, where $U$ and $V$ are the subsets in the spaces $R^m$ and $R^r$, respectively. We denote by $|\cdot|$ the Euclidean norm of the vector, and by $\|\cdot\|$ we denote the norm of the matrix consistent with the norm of the vector.

Controls of $u=u(t)=(u_1(t),u_2(t),\dots,u_m(t))$ and $v=v_i=(v_{i1},v_{i2},\dots,v_{ir})$ will be considered admissible for problems (0.1)–(0.3) if:

(a1) the function $u(t)$ is measurable and locally integrated at $t \geq 0$;

(a2) $u(t) \in U$, $t \geq 0$;

(a3) for every $u(t)$ there exists a constant $u_0 \in U$ such that $u(t) \to u_0$ for $t \to \infty$ uniformly for all controls, i.e. for arbitrary $\delta > 0$ there exists a constant $T_0 > 0$, independent of $u(t)$, $u_0$, such that for all $t \geq T_0$ the inequality $|u(t) - u_0| < \delta$ holds;

(a4) for each sequence of vectors $v_i$ there exists $v_0 \in V$ such that $v_i \to v_0$, $i \to \infty$ uniformly for all controls, i.e. for arbitrary $\delta > 0$ there exists a constant $N_0$, independent of $v_i$, $v_0$, such that for all $i \geq N_0$ the inequality $|v_i - v_0| < \delta$ is satisfied;

(a5) condition $|J_\varepsilon(u,v)| < \infty$ holds for functional (0.3).

Note that conditions (a3) and (a4) are obviously satisfied if there exist a function $\varphi(t) \to 0$, and a sequence $\varphi(t) \to 0$, $t \to \infty$ which are independents of $u(t)$ and $v_i$, respectively, such that $|u(t) - u_0| \leq \varphi(t)$, $\quad |v_i - v_0| < a_i$. Condition (a3) for control, first appeared in M. M. Moiseyev [3], when applying the method of averaging to practical problems. In this monograph, such controls are called asymptotically constant.

We denote the set of admissible controls of problems (0.1), (0.2) and (0.1)–(0.3) by $F_1$ and $F_2$, respectively. In this case,

$$J_\varepsilon^1 = \inf_{(u,v)\in F_1} J_\varepsilon^1(u,v)$$

and

$$J_\varepsilon^2 = \inf_{(u,v)\in F_2} J_\varepsilon^2(u,v).$$

Denote by $x_\varepsilon(t,u,v)$ the solution of the Cauchy problem corresponding to the admissible control $(u,v)$. The triple $(x_\varepsilon^*(t,u,v), u_\varepsilon^*, v_\varepsilon^*)$ is optimal for problems (0.1)–(0.3) if $(u_\varepsilon^*, v_\varepsilon^*)$ is an admissible pair and $J_\varepsilon^1(u_\varepsilon^*, v_\varepsilon^*) = J_\varepsilon^1$ for functional (0.2), or $J_\varepsilon^2(u_\varepsilon^*, v_\varepsilon^*) = J_\varepsilon^2$ for functional (0.3).

Let the averaging conditions be satisfied:

(a6) there are limits uniformly across $t \geq 0$, $x \in D$, $u \in U$, $v \in V$:

$$\lim_{s\to\infty} \frac{1}{s} \int_t^{s+t} X(\tau,x,u)\,d\tau = X_0(x,u), \tag{0.4}$$

$$\lim_{s\to\infty} \frac{1}{s} \sum_{t<t_i(x)<s+t} I_i(x,v) = I_0(x,v), \tag{0.5}$$

$$\lim_{s\to\infty} \frac{1}{s} \int_t^{s+t} \Phi(\tau,x,u)\,d\tau = \Phi_0(x,u), \tag{0.6}$$

$$\lim_{s\to\infty} \frac{1}{s} \sum_{t<t_i(x)<s+t} \Psi_i(x,v) = \Psi_0(x,v). \tag{0.7}$$

With respect to the moments of impulse action, we will assume that there exists a constant $C > 0$ such that for $t \geq 0$, $x \in D$

$$\sum_{t<t_i(x)<s+t} I_i \leq Cs. \tag{0.8}$$

We will put averaged tasks in accordance with the problems of optimal control (0.1)–(0.3)

$$\dot{y} = \varepsilon[X_0(y,u_0) + I_0(y,v_0)], \quad y(0) = x_0, \tag{0.9}$$

$$\overline{J}_\varepsilon^{\,1}(u_0,v_0) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} \big[\Phi_0(y(t),u_0) + \Psi_0(y(t),v_0)\big]\,dt \longrightarrow \inf, \tag{0.10}$$

$$\overline{J}_\varepsilon^{\,2}(u_0,v_0) = \varepsilon \int\limits_0^{\infty} e^{-\gamma t} L(t,y(t))\,dt \longrightarrow \inf, \tag{0.11}$$

where $u_0 \in U$, $v_0 \in V$ are already constant vectors. These tasks are much simpler than the original ones because they are problems of optimal control for systems of ordinary differential equations. Denote by analogy as in the case of initial problems $\overline{J}_\varepsilon^{\,1} = \inf\limits_{(u_0,v_0)\in F_1} \overline{J}_\varepsilon^{\,1}(u_0,v_0)$ and $\overline{J}_\varepsilon^{\,2} = inf_{(u,v)\in F_2} \overline{J}_\varepsilon^{\,2}(u_0,v_0)$.

The main result is obtained which states that the optimal control $(u_0^*(\varepsilon), v_0^*(\varepsilon))$ of averaged problems is $\eta$-optimal for the initial problems, namely, for arbitrary $\eta > 0$ there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ the inequalities:

$$\big|J_\varepsilon^1(u_0^*(\varepsilon), v_0^*(\varepsilon)) - J_\varepsilon^1\big| < \eta, \quad \big|J_\varepsilon^2(u_0^*(\varepsilon), v_0^*(\varepsilon)) - J_\varepsilon^2\big| < \eta$$

are satisfied.

It is known that the averaging method is one of the most common methods of analyzing nonlinear dynamic systems. For ordinary differential equations, this method was substantiated by M. M. Bogolyubovym [1]. The validation of this method for systems with impulse action in the general form was first obtained in [6]. We also note the works [7, 9], where the results of [6] have been further developed.

The averaging method also proved to be effective for solving problems of optimal control. A number of papers are devoted to this question (see, for example, [5], where there is an extensive bibliography). In [4] developed a different approach as for to applying the averaging method to tasks of optimal control, namely, considering the control function $u$ as a parameter, was averaging over by time, that clearly included in the right-parts sides of the system.

In this paper, the approach under consideration is applied to the problems of optimal control of impulse systems with non-fixed moments of impulse actions. Such problems with the application of the principle of maximum were previously studied in [8].

This paper describes the problem formulation and reviews the literature, gives strict formulation of the problem, and presents the main results obtained when solving the problems under consideration.

## 1 Statement of the problem and formulation of the main results

In what follows, we consider the following conditions for problems (0.1)–(0.3) and their corresponding averaged problems (0.9)–(0.11):

2.1. The functions $X$, $I_i$, $\Phi$, $\Psi_i$, $L$ are uniformly continuous on the set of variables at $t \geq 0$, $x \in D$, $u \in U$, $v \in V$, evenly at $i = 1, 2, \dots$ .

2.2. There is a positive constant $M$ such that

$$\Big|\frac{\partial t_i(x)}{\partial x}\Big| + |X(t,x,u)| + |\Phi(t,x,u)| + |\Psi_i(x,v)| + |I_i(x,v)| \leq M,$$

for $t \geq 0$, $x \in D$, $u \in U$, $v \in V$, $i = 1, 2, \dots$ .

2.3. There is a positive constant $K$ such that

$$|X(t,x,u)-X(t,x_1,u)|+|I_i(x,v)-I_i(x_1,v)|+|\Phi(t,x,u)-\Phi(t,x_1,u)| \\ +|\Psi_i(x,v)-\Psi_i(x_1,v)|+\Big|\frac{\partial t_i(x)}{\partial x}-\frac{\partial t_i(x_1)}{\partial x}\Big|\le K|x-x_1|,\quad \Big|\frac{\partial t_i(x)}{\partial x}\Big|\le K$$

for $t\ge 0$, $x,x_1\in D$, $i=1,2,\dots$, $u\in U$, $v\in V$.

2.4. Condition (a5) is satisfied.

2.5. The averaged Cauchy problem (0.9) has the solution $y(\varepsilon t)=y(\varepsilon t,x_0,u_0,v_0), y(0,x_0,u_0,v_0)=x_0$, which for $\varepsilon=1$ belongs to $D$ for $t\in[0,T]$ together with some own $\rho$-circle (independent of $u_0$, $v_0$) and the inequality

$$\frac{\partial t_i(y(\varepsilon t))}{\partial x}I_i(y(\varepsilon t),v)\le\beta<0$$

holds when $t_i'<t<t_i''$, $v\in V$, or

$$\frac{\partial t_i(x)}{\partial x}\equiv 0.$$

Here

$$t_i'=\inf_{x\in D}t_i(x),\quad t_i''=\sup_{x\in D}t_i(x),\quad i=\overline{1,l},\quad t_l<\frac{T}{\varepsilon}<t_{l+1}.$$

The following theorem is on the connection between problems of optimal control on finite time intervals.

**Theorem 1.1** ([2])**.** *Let conditions* 2.1–2.5 *be satisfied and there be an optimal control* $(u_0^*(\varepsilon),v_0^*(\varepsilon))$ *of the averaged problem* $(0.9),(0.10)$ *for* $0<\varepsilon\le\varepsilon_0$*. Then for arbitrary* $\eta>0$ *there exists* $\varepsilon_1=\varepsilon_1(\eta,\varepsilon_0)>0$ *such that for all* $\varepsilon\in(0,\varepsilon_1)$ *the following conditions hold:*

(1) $J_\varepsilon^1>-\infty$;

(2) *the inequality holds*

$$|J_\varepsilon^1(u_0^*(\varepsilon),v_0^*(\varepsilon))-J_\varepsilon^1|\le\eta. \tag{1.1}$$

**Remark 1.1.** If the conditions of Theorem 1.1 state that the sets of admissible controls $U$ and $V$ are compact, then the optimal control $(u_0^*(\varepsilon),v_0^*(\varepsilon))$ of the averaged problem exists.

Indeed, the solution of the averaged problem (0.9) extends to the interval $[0,\frac{T}{\varepsilon}]$. Conditions of Theorem 1.1 imply that $y(t,u_0,v_0)$ is a continuous function of the parameters $u_0$ and $v_0$, therefore, Lebesgue's theorem on majorized convergence also implies the continuity of $\overline{J}_\varepsilon^1(u_0,v_0)$ over $u_0$ and $v_0$. The statement of Remark 1.1 is now a consequence of the Weierstrass theorem.

**Remark 1.2.** If $X_0(y,u_0)+I_0(y,v_0)$, $\Phi_0(y,u_0)+\Psi_0(y,v_0)$ are continuous differentiated functions, then problem $(0.9),(0.10)$ is a smooth finite-dimensional extremal problem.

Consider the problem of optimal control on the axis, for this system (0.9) we write at "slow time": $\tau=\varepsilon t$:

$$\frac{dy}{d\tau}=\big[X_0(y,u_0)+I_0(y,v_0)\big],\quad y(0)=x_0. \tag{1.2}$$

**Theorem 1.2** ([2]). *Let the conditions* 2.1–2.5 *hold, and let the solution $y(\tau)=y(\tau,x_0,u_0,v_0)$ of the Cauchy problem* (1.2) *be uniformly asymptotically stable at $\tau_0$, $u_0$ and $v_0$, and belong to the domain $D$ at $\tau\geq 0$ together with its some $p$-circle (independent of $u_0$, $v_0$), and the inequalities $\frac{\partial t_i(x)}{\partial x}I_i(x)\leq\beta<0$ (or $\frac{\partial t_i(x)}{\partial x}\equiv 0$) hold for all $i=1,2,\dots$ and $x$ from some $\rho_0$-circle of the solution $y(\tau)$.*

*Then, if there is an optimal control $(u_0^*(\varepsilon),v_0^*(\varepsilon))$ for $\varepsilon\in(0,\varepsilon_0]$ of the averaged problem* (0.9), (0.11)*, then for arbitrary h> 0 there is $\varepsilon_1=\varepsilon_1(\varepsilon_0,\eta)>0$ such that*

(1) *for arbitrary $\varepsilon\in(0,\varepsilon_1)$, it holds $|J_\varepsilon^2|<\infty$;*

(2) *the inequality $|J_\varepsilon^2(u_0^*(\varepsilon),v_0^*(\varepsilon))-J_\varepsilon^2|\leq\eta$ holds.*

**Remark 1.3.** If under Theorem 1.2 the sets of admissible controls are compact, then optimal control of the averaged problem (0.9), (0.10) exists.

This observation follows from a continuous dependence on the parameters at each finite interval of the solution $y(t,u_0,v_0)$ and Lebesgue, Weierstrass theorems. The proof is based on the corresponding result by A. M. Samoilenko from [6, Theorem 1] for unmanaged impulse systems.

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