# Initial Value Method in Boundary Value Problems for Systems of Two-Term Fractional Differential Equations at Resonance

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## 1 Introduction

Let $T > 0$ be given, $J = [0,T]$ and $X = C(J) \times C(J)$.

We investigate the system of fractional differential equations

$$\left.\begin{aligned} {}^c\!D^{\alpha}u(t) + p(t)\,{}^c\!D^{\alpha_1}u(t) = f(t,u(t),v(t)),\\ {}^c\!D^{\beta}v(t) + q(t)\,{}^c\!D^{\beta_1}v(t) = g(t,u(t),v(t)),\end{aligned}\right\} \tag{1.1}$$

where $0 < \alpha_1 < \alpha \le 1$, $0 < \beta_1 < \beta \le 1$, $p, q \in C(J)$, $f, g \in C(J \times \mathbb{R}^2)$ and ${}^c\!D$ denotes the Caputo fractional derivative.

Let $\mathcal{K}, \mathcal{R} : C(J) \to \mathbb{R}$ be functionals given as

$$\mathcal{K}x = \sum_{k=1}^{m_1} c_k x(\rho_k), \qquad \mathcal{R}x = \sum_{k=1}^{m_2} d_k x(\xi_k),$$

where $m_j \in \mathbb{N}$ or $m_j = \infty$, $j = 1,2$, $\{\rho_k\}_{k=1}^{m_1} \subset (0,T]$, $\{\xi_k\}_{k=1}^{m_2} \subset (0,T]$ are increasing sequences and $c_k > 0$, $d_k > 0$, $\sum\limits_{k=1}^{m_1} c_k = 1$, $\sum\limits_{k=1}^{m_2} d_k = 1$.

Together with system (1.1) we study the boundary condition

$$(u(0), v(0)) = (\mathcal{K}u, \mathcal{R}v). \tag{1.2}$$

**Definition 1.1.** We say that $(u,v) : J \to \mathbb{R}^2$ is *a solution of system* (1.1) if $(u,v), ({}^c\!D^{\alpha}u, {}^c\!D^{\beta}v) \in X$ and $(u,v)$ satisfies (1.1) for $t \in J$. A solution $(u,v)$ of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

Since each constant vector-function $(u,v)$ on the interval $J$ is a solution of problem ${}^c\!D^{\alpha}u + p(t)\,{}^c\!D^{\alpha_1}u = 0$, ${}^c\!D^{\beta}v + q(t)\,{}^c\!D^{\beta_1}v = 0$, (1.2), problem (1.1), (1.2) is at resonance.

We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative [1,2].

*The Riemann–Liouville fractional integral* $I^{\gamma}x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^{\gamma}x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\,\mathrm{d}s,$$

where *Gamma* is the Euler gamma function. $I^0$ is the identical operator.

*The Caputo fractional derivative* ${}^{c}D^{\gamma}x$ of order $\gamma \in (0,1)$ of a function $x : J \to \mathbb{R}$ is given as

$$ {}^{c}D^{\gamma}x(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t}\, I^{1-\gamma}\,(x(t)-x(0)). $$

If $\gamma = 1$, then ${}^{c}D^{\gamma}x(t) = x'(t)$.

The special case of (1.1) (for $\alpha = 1$, $\beta = 1$) is the system of generalized Basset fractional differential equations [3]

$$ \left.\begin{aligned} u'(t) + p(t){}^{c}D^{\alpha_1}u(t) &= f(t,u(t),v(t)),\\ v'(t) + q(t){}^{c}D^{\beta_1}v(t) &= g(t,u(t),v(t)). \end{aligned}\right\} $$

The special cases of (1.2) are the periodic condition

$$ (u(0),v(0)) = (u(T),v(T)) $$

and the infinite-point boundary condition

$$ (u(0),v(0)) = \Big(\sum_{k=1}^{\infty} c_k u(\rho_k), \sum_{k=1}^{\infty} d_k v(\xi_k)\Big). $$

We will work with the following conditions for the functions $p$, $q$, $f$ and $g$ in (1.1):

$(H_1)$ There exist $D,H,K,L \in \mathbb{R}$, $D < H$, $K < L$, such that

$$ \begin{aligned} &f(t,D,y) > 0, \quad f(t,H,y) < 0 \ \text{ for } t \in J,\ y \in [K,L],\\ &g(t,x,K) > 0, \quad f(t,x,L) < 0 \ \text{ for } t \in J,\ x \in [D,H]. \end{aligned} $$

$(H_2)$ $p(t) \ge 0$ and $q(t) \ge 0$ for $t \in J$.

The aim of this paper is to discuss the existence of solutions to problem (1.1), (1.2). The existence results are proved by the following procedure. By the combination of initial value method [4] with the maximum principle for the Caputo fractional derivative [4] and the Schaefer fixed point theorem we first prove that for each $(c_1,c_2) \in [D,H] \times [K,L]$ there exists a solution $(u,v)$ of system (1.1) on the interval $J$ satisfying the initial condition $(u(0),v(0)) = (c_1,c_2)$. Then we discuss the set $\mathcal{C}$ of all such solutions and show that $\mathcal{C}$ is a compact metric space. Assuming that $(u(0),v(0)) \neq (\mathcal{K}u,\mathcal{R}v)$ for all $(u,v) \in \mathcal{C}$ we obtain a contradiction by the study of some compact subsets of $\mathcal{C}$.

## 2 Initial value problem

For $r \in C(J)$ and $\gamma \in (0,1)$, let $\Lambda_{r,\gamma} : C(J) \to C(J)$ be defined as

$$ \Lambda_{r,\gamma}x(t) = -r(t)I^{\gamma}x(t) $$

and $\Lambda^0_{r,\gamma}$ be the identical operator on $C(J)$. For $n \in \mathbb{N}$, let $\Lambda^n_{r,\gamma} = \underbrace{\Lambda_{r,\gamma} \circ \Lambda_{r,\gamma} \circ \cdots \circ \Lambda_{r,\gamma}}_{n}$ be $n$th iteration of $\Lambda_{r,\gamma}$. Let $\mathcal{D}_{r,\gamma} : C(J) \to C(J)$ be an operator defined as

$$ \mathcal{D}_{r,\gamma}x(t) = \sum_{n=0}^{\infty} \Lambda^n_{r,\gamma}x(t). $$

Let $(H_1)$ hold. Let

$$\eta(x)=\begin{cases}H & \text{if } x>H,\\ x & \text{if } x\in[D,H],\\ D & \text{if } x<D,\end{cases}\qquad \rho(y)=\begin{cases}L & \text{if } y>L,\\ y & \text{if } y\in[K,L],\\ K & \text{if } y<K,\end{cases}$$

and $f^*,g^*:J\times\mathbb{R}^2\to\mathbb{R}$ be given as

$$f^*(t,x,y)=f(t,\eta(x),\rho(y)),\quad g^*(t,x,y)=g(t,\eta(x),\rho(y)).$$

Then $f^*,g^*\in C(J\times\mathbb{R}^2)$ are bounded and

$$\left.\begin{array}{ll} f^*(t,x,y)>0 \ \text{ if } x<D,\, y\in\mathbb{R}, & f^*(t,x,y)<0 \ \text{ if } x>H,\, y\in\mathbb{R},\\ g^*(t,x,y)>0 \ \text{ if } x\in\mathbb{R},\, y<K, & g^*(t,x,y)<0 \ \text{ if } x\in\mathbb{R},\, y>L,\end{array}\right\}$$

for $t\in J$. Let operators $\mathcal{F},\mathcal{G}:X\to C(J)$ be the Nemytskii operators associated to $f^*$, $g^*$,

$$\mathcal{F}(x,y)(t)=f^*(t,x(t),y(t)),\quad \mathcal{G}(x,y)(t)=g^*(t,x(t),y(t)),$$

and $\mathcal{A},\mathcal{B}:C(J)\to C(J)$,

$$\mathcal{A}x(t)=\mathcal{D}_{p,\alpha-\alpha_1}x(t),\quad \mathcal{B}x(t)=\mathcal{D}_{q,\beta-\beta_1}x(t),$$

where $p$, $q$, $\alpha$, $\alpha_1$, $\beta$ and $\beta_1$ are from (1.1).

We now consider the fractional initial value problem

$$\left.\begin{array}{r} {}^cD^{\alpha}u(t)+p(t){}^cD^{\alpha_1}u(t)=f^*(t,u(t),v(t)),\\ {}^cD^{\beta}v(t)+q(t){}^cD^{\beta_1}v(t)=g^*(t,u(t),v(t)),\end{array}\right\} \tag{2.1}$$

$$(u(0),v(0))=(c_1,c_2),\ \ (c_1,c_2)\in\mathbb{R}^2. \tag{2.2}$$

Let an operator $\mathcal{Q}:X\to X$ be defined by the formula

$$\mathcal{Q}(x,y)=(\mathcal{Q}_1(x,y),\mathcal{Q}_2(x,y)),$$

where $Q_j:X\to C(J)$,

$$\mathcal{Q}_1(x,y)(t)=c_1+I^{\alpha}\mathcal{A}\mathcal{F}(x,y)(t),\quad \mathcal{Q}_2(x,y)(t)=c_2+I^{\beta}\mathcal{B}\mathcal{G}(x,y)(t),$$

and $c_1$, $c_2$ are from (2.2).

The following result gives the relation between solutions of problem (2.1), (2.2) and fixed points of $\mathcal{Q}$.

**Lemma 2.1.** *Let $(H_1)$ hold. Then $(u,v)$ is a fixed point of $\mathcal{Q}$ if and only if $(u,v)$ is a solution of problem* (2.1), (2.2).

The existence results for problems (2.1), (2.2) and (1.1), (2.2) are stated in the following two lemmas.

**Lemma 2.2.** *Let $(H_1)$ hold. Then there exists at least one solution of problem* (2.1), (2.2).

Let $\Delta=[D,H]\times[K,L]$, where $D,H,K$ and $L$ are from $(H_2)$.

**Lemma 2.3.** *Let $(H_1)$, $(H_2)$ hold and let $(c_1,c_2)\in\Delta$. Then problem* (1.1), (2.2) *has at least one solution and all its solutions $(u,v)$ satisfy*

$$D<u(t)<H,\ \ K<v(t)<L \ \textit{ for } \ t\in(0,T]. \tag{2.3}$$

## 3 Existence result for problem (1.1), (1.2)

**Theorem 3.1.** *Let $(H_1)$ and $(H_2)$ hold. The problem* (1.1), (1.2) *has at least one solution $(u,v)$ and*

$$D < u(t) < H, \ \ K < v(t) < L \ \ \text{for} \ \ t \in J. \tag{3.1}$$

**Sketch of proof.** Having in mind Lemma 2.3, for $(c_1, c_2) \in \Delta$ let $\mathcal{C}_{(c_1,c_2)}$ be the set of all solutions to problem (1.1), (2.2). Let

$$\mathcal{C} = \bigcup_{(c_1,c_2)\in\Delta} \mathcal{C}_{(c_1,c_2)}.$$

Then for each $(u,v) \in \mathcal{C}$ the equalities

$$u(t) = u(0) + I^{\alpha}\mathcal{A}\mathcal{F}(u,v)(t), \quad v(t) = v(0) + I^{\beta}\mathcal{B}\mathcal{G}(u,v)(t), \ \ t \in J,$$

and inequality (2.3) hold. We can prove that $\mathcal{C}$ is a compact metric space equipped with the metric

$$\rho((u,v),(u_1,v_1)) = \max\big\{|u(t)-u_1(t)| : \ t \in J\big\} + \max\big\{|v(t)-v_1(t)| : \ t \in J\big\}.$$

Assume to the contrary that

$$|u(0) - \mathcal{K}u| + |v(0) - \mathcal{R}v| > 0 \ \ \text{for} \ \ (u,v) \in \mathcal{C}, \tag{3.2}$$

where $\mathcal{K}, \mathcal{R}$ are from the boundary condition (1.2). Condition (3.2) is equivalent to

$$(u,v) \in \mathcal{C} \implies \begin{cases} \text{either } u(0) - \mathcal{K}u = 0 \text{ and } v(0) - \mathcal{R}v \neq 0 \\ \text{or } u(0) - \mathcal{K}u \neq 0 \text{ and } v(0) - \mathcal{R}v = 0. \end{cases} \tag{3.3}$$

Keeping in mind (3.3), let

$$\mathcal{P}_1 = \big\{(u,v) \in \mathcal{C} : \ u(0) = \mathcal{K}u, \ v(0) - \mathcal{R}v \neq 0\big\},$$
$$\mathcal{P}_2 = \big\{(u,v) \in \mathcal{C} : \ u(0) - \mathcal{K}u \neq 0, \ v(0) = \mathcal{R}v\big\}.$$

Then $\mathcal{C} = \mathcal{P}_1 \cup \mathcal{P}_2$ and $\mathcal{P}_1 \cap \mathcal{P}_2 = \varnothing$ and we can prove that $\mathcal{P}_1$, $\mathcal{P}_2$ are nonvoid compact subsets of $\mathcal{C}$. Hence the compact metric space $\mathcal{C}$ is the union of nonvoid, mutually disjoint compact subsets $\mathcal{P}_1, \mathcal{P}_2$, which is impossible. As a result assumption (3.2) is false. Consequently, problem (1.1), (1.2) has a solution $(u,v)$.

It remains to prove that $(u,v)$ satisfies inequality (3.1). We know that $(u,v)$ satisfies inequality (2.3). Assume, for example, that $v(0) = K$. Since $v > K$ on $(0,T]$, we have

$$v(0) - \mathcal{R}v = v(0) - \sum_{k=1}^{m_2} d_j v(\xi_j) < K - K\sum_{k=1}^{m_2} d_j = K - K = 0,$$

which contradicts $v(0) - \mathcal{R}v = 0$. Hence $v > K$ on $J$. $\square$

**Example 3.1.** Let $r, l, p, q \in C(J)$, $r > 1$, $l > 0$, and let $\rho \geq 1$. Then the functions $f(t,x,y) = r(t) - e^{x} + e^{-y}$, $g(t,x,y) = l(t) + x - |y|^{\rho}$ satisfy condition $(H_1)$ for $D = 0$, $H = \ln(2 + \|r\|)$, $K = 0$ and $L = \sqrt[q]{1 + \|l\| + \ln(2 + \|r\|)}$. Applying Theorem 3.1, the system

$$\left.\begin{aligned} {}^{c}D^{\alpha}u + |p(t)|{}^{c}D^{\alpha_1}u &= r(t) - e^{u} + e^{-v}, \\ {}^{c}D^{\beta}v + |q(t)|{}^{c}D^{\beta_1}v &= l(t) + u - |v|^{\rho} \end{aligned}\right\}$$

has a solution $(u,v)$ satisfying the boundary condition (1.2) and

$$0 < u(t) < \ln(2 + \|r\|), \quad 0 < v(t) < \sqrt[q]{1 + \|l\| + \ln(2 + \|r\|)}\,, \ \ t \in J.$$

## References


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