Asymptotic Representations for Solutions of Non-Linear Systems of Ordinary Differential Equations

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We consider the system of differential equations

$$y_i' = f_i(t, y_1, \dots, y_n), \ \ i = \overline{1,n}, \tag{1}$$

where $f_i : [a,\omega[\times \prod\limits_{i=1}^{n} \Delta(Y_i^0) \to \mathbb{R}$, $i = \overline{1,n}$, are continuous functions, $-\infty < a < \omega \le +\infty$[1], $\Delta(Y_i^0)$, $i \in \{1, \dots, n\}$ is one-sided neighborhood of Y_i^0, Y_i^0 equals either 0 or $\pm\infty$.

Definition 1. A solution $(y_i)_{i=1}^n$ of system (1) is called $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution if it is defined on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions

$$y_i(t) \in \Delta(Y_i^0) \ \text{ while } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y_i(t) = Y_i^0, \tag{2}$$

$$\lim_{t \uparrow \omega} \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} = \Lambda_i \ \ (i = \overline{1, n-1}). \tag{3}$$

System (1) was considered in T. A. Chanturia's works [1, 2]. In these works, T. A. Chanturia obtained results about existence of proper, singular and oscillating solutions of system (1). These results are especially effective for cyclic systems.

In [3–5, 7, 8], the asymptotics for $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solutions for cyclic differential equations systems of the following form were considered

$$y_i' = \alpha_i p_i(t) \varphi_{i+1}(y_{i+1}) \ \ (i = \overline{1,n})^2,$$

where $\alpha_i \in \{-1, 1\}$ $(i = \overline{1,n})$, $p_i : [a, \omega[\to]0, +\infty[$ $(i = \overline{1,n})$ are continuous functions, $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = \overline{1,n})$ are continuously differentiable functions and satisfy conditions

$$\lim_{\substack{y_i \to Y_i^0 \\ y_i \in \Delta(Y_i^0)}} \frac{y_i \varphi_i'(y_i)}{\varphi_i(y_i)} = \sigma_i \ \ (i = \overline{1,n}), \quad \prod_{i=1}^{n} \sigma_i \neq 1.$$

Assume that the definition of $\mathcal{P}_\omega(\Lambda_1, \dots, \Lambda_{n-1})$-solution does not give the direct connection between the first and the $n-$th components of this solution. In order to establish this connection, we define the following functions

$$\lambda_i(t) = \frac{y_i(t) y_{i+1}'(t)}{y_i'(t) y_{i+1}(t)} \ \ (i = \overline{1,n}).$$

[1]For $\omega = +\infty$ consider $a > 0$.

[2]Here and in the sequel, all functions and parametres with the subscript $n+1$ are assumed to coincide with those with the subscript 1.

We proceed and show that

$$\lambda_n(t)=\frac{y_n(t)y_1'(t)}{y_n'(t)y_1(t)}=\frac{y_n(t)y_{n-1}'(t)}{y_n'(t)y_{n-1}(t)}\cdot\frac{y_{n-1}(t)y_{n-2}'(t)}{y_{n-1}'(t)y_{n-2}(t)}\cdots\frac{y_2(t)y_1'(t)}{y_2'(t)y_1(t)}=\frac{1}{\lambda_1(t)\cdots\lambda_{n-1}(t)}\,. \tag{4}$$

From (3) it follows that $\lim\limits_{t\uparrow\omega}\lambda_i(t)=\Lambda_i$ $(i=\overline{1,n-1})$. Therefore, if there are zeros among Λ_i $(i=\overline{1,n-1})$ from (4), we obtain

$$\Lambda_n=\lim_{t\uparrow\omega}\lambda_n(t)=\pm\infty.$$

In particular, it is evident that the case, when among all Λ_i $(i=1,\dots,n-1)$ there is a single $\pm\infty$, while all others are real numbers different from zero, could be transformed into the case described in this work. This transformation is carried out by cyclic redesignation of variables, functions and constants. For instance, if $\Lambda_l=\pm\infty$ $(l\in\{1,\dots,n-1\})$, the indices are redesignated as follows

$$l\longrightarrow n,\ l+1\longrightarrow 1,\ \dots,\ n\longrightarrow n-l,\ 1\longrightarrow n-l+1,\ \dots,\ l-1\longrightarrow n-1.$$

It is obvious that $\Lambda_i=0$ when $i=n-l$.

Further, we introduce auxiliary notation.

First, if

$$\mu_i=\begin{cases}1 & \text{as } Y_i^0=+\infty, \text{ or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is right neighborhood of } 0,\\ -1 & \text{as } Y_i^0=-\infty, \text{ or } Y_i^0=0 \text{ and } \Delta(Y_i^0) \text{ is left neighborhood of } 0,\end{cases}$$

it is obvious that μ_i $(i=\overline{1,n})$ determine the signs of the components of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solution in some left neighborhood of ω.

The existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions of system (1) for fixed values of $\Lambda_i\in\mathbb{R}$ $(\prod\limits_{i=1}^{n-1}\Lambda_i=0)$, $i=\overline{1,n-1}$, and their asymptotics as $t\uparrow\omega$ will be explored when this system is in a certain way close to a cyclic one with regularly varying non-linearities.

Definition 2. We say that system (1) satisfies the condition $N(\Lambda_1,\dots,\Lambda_{n-1})$, where $\Lambda_i\in\mathbb{R}$, $i=\overline{1,n-1}$, if for any $k\in\{1,\dots,n\}$ there exist a number $\alpha_k\in\{-1,1\}$, a continuous function $p_k:[a,\omega[\,\to\,]0,+\infty[$ and continuous regularly varying $\varphi_{k+1}:\Delta(Y_{k+1}^0)\to\,]0;+\infty[$ of σ_{k+1} orders (when $y_{k+1}\to Y_{k+1}^0$) which admit the following representation for any functions $y_i:[a,\omega[\,\to\Delta(Y_i^0)$, $i=\overline{1,n}$, satisfying conditions (2), (3):

$$f_k(t,y_1(t),\dots,y_n(t))=\alpha_k p_k(t)\varphi_{k+1}(y_{k+1}(t))[1+o(1)] \ \text{ when } \ t\uparrow\omega. \tag{5}$$

Since functions φ_i $(i=\overline{1,n})$ are regularly varying as $z\to Y_i^0$ of σ_i orders, they admit the following representation (see [6]):

$$\varphi_i(y_i)=|y_i|^{\sigma_i}\theta_i(y_i)\ \ (i=\overline{1,n}), \tag{6}$$

where $\theta_i:\Delta(Y_i^0)\to\,]0;+\infty[$ $(i=\overline{1,n})$ are slowly varying functions as $z\to Y_i^0$.

Having supposed that system (1) for certain Λ_i, $i\in\{1,\dots,n-1\}$, satisfies the condition $N(\Lambda_1,\dots,\Lambda_{n-1})$ and $\prod\limits_{k=1}^{n}\sigma_k\neq 1$ (for orders σ_k, $k=\overline{1,n}$ of functions φ_k), we introduce auxiliary designation.

We denote sets

$$\mathfrak{I}=\big\{i\in\{1,\dots,n-1\}:\ 1-\Lambda_i\sigma_{i+1}\neq 0\big\},\quad \overline{\mathfrak{I}}=\{1,\dots,n-1\}\setminus\mathfrak{I}$$

and suppose that $1-\Lambda_{n-1}\sigma_n \neq 0$.

By taking into account the fact that $n-1 \in \mathfrak{I}$, we denote auxiliary functions I_i, Q_i $(i=1,\dots,n)$ and non-zero constants β_i $(i=1,\dots,n)$, supposing that

$$I_i(t) = \begin{cases} \displaystyle\int_{A_i}^{t} p_i(\tau)\,d\tau & \text{for } \ i \in \mathfrak{I}, \\ \displaystyle\int_{A_i}^{t} p_i(\tau) I_{i+1}(\tau)\,d\tau & \text{for } \ i \in \overline{\mathfrak{I}}, \\ \displaystyle\int_{A_n}^{t} p_n(\tau) q_n(\tau)\,d\tau & \text{for } \ i = n, \end{cases}$$

$$\beta_i = \begin{cases} 1-\Lambda_i\sigma_{i+1} & \text{if } \ i \in \mathfrak{I}, \\ \beta_{i+1}\Lambda_i & \text{if } \ i \in \overline{\mathfrak{I}}, \\ 1-\displaystyle\prod_{k=1}^{n}\sigma_k & \text{if } \ i=n, \end{cases} \qquad Q_i(t) = \begin{cases} \alpha_i\beta_i I_i(t) & \text{for } \ i \in \mathfrak{I}\cup\{n\}, \\ \dfrac{\alpha_i\beta_i I_i(t)}{I_{i+1}(t)} & \text{for } \ i \in \overline{\mathfrak{I}}, \end{cases}$$

where each limit of integration $A_i \in \{\omega, a\}$ $(i \in \{1,\dots,n-1\})$, $A_n \in \{\omega, b\}$ ($b \in [a,\omega[$) is chosen in such a way that its corresponding integral I_i aims either to zero, or to ∞ as $t \uparrow \omega$,

$$q_n(t) = \theta_1\big(\mu_1|I_1(t)|^{\frac{1}{\beta_1}}\big)|Q_{n-1}(t)|^{\prod\limits_{k=1}^{n-1}\sigma_k} \prod_{k=1}^{n-2}\Big|Q_k(t)\theta_{k+1}\big(\mu_{k+1}|I_{k+1}(t)|^{\frac{1}{\beta_{k+1}}}\big)\Big|^{\prod\limits_{i=1}^{k}\sigma_i}.$$

In addition, we introduce numbers

$$A_i^* = \begin{cases} 1 & \text{if } \ A_i = a, \\ -1 & \text{if } \ A_i = \omega \end{cases} \quad (i=1,\dots,n-1), \quad A_n^* = \begin{cases} 1 & \text{if } \ A_n = b, \\ -1 & \text{if } \ A_n = \omega. \end{cases}$$

These numbers enable us to define the signs of functions I_i $(i=1,\dots,n-1)$ on the interval $]a,\omega[$ and the sign of function I_n on the interval $]b,\omega[$.

Definition 3. We say that the function φ_k $(k \in \{1,\dots,n\})$ satisfies the condition **S** if for any continuously differentiable function $l : \Delta(Y_k^0) \to]0,+\infty[$ with the property

$$\lim_{\substack{z \to Y_k^0 \\ z \in \Delta(Y_k^0)}} \frac{z\,l'(z)}{l(z)} = 0,$$

the function θ_k (defined in (6)) admits the asymptotic representation

$$\theta_k(zl(z)) = \theta_k(z)[1+o(1)] \ \text{ when } \ z \to Y_k^0 \ \ (z \in \Delta(Y_k^0)).$$

For instance, **S** – condition is obviously satisfied by functions φ_k of the following type

$$\varphi_k(y_k) = |y_k|^{\sigma_k}|\ln y_k|^{\gamma_1}, \quad \varphi_k(y_k) = |y_k|^{\sigma_k}|\ln y_k|^{\gamma_1}|\ln|\ln y_k||^{\gamma_2},$$

where $\gamma_1, \gamma_2 \neq 0$. **S** – condition is also satisfied by functions φ_k which include functions θ_k that have the eventual limit as $y_k \to Y_k^0$. **S** – condition is also satisfied by many other functions.

By means of introduced designations, we will establish the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$-solutions for (1).

Theorem. *Let system* (1) *satisfy* $N(\Lambda_1,\dots,\Lambda_{n-1})$*-condition and* $\Lambda_i\in\mathbb{R}$ $(i=\overline{1,n-1})$ *include those equal zeros,* $n-1\in\mathfrak{I}$ *and* $m=\max\{i\in\mathfrak{I}:\ \Lambda_i=0\}$*. Let also functions* φ_k $(k=\overline{1,n-1})$*, defined in* (5)*, satisfy* $\mathbf{S}$*-condition. Then for the existence of* $\mathcal{P}_\omega(\Lambda_1,\dots,\Lambda_{n-1})$*-solutions of system* (1) *it is necessary, and if the algebraic equation*

$$(1+\lambda)\prod_{j=m+1}^{n-1}(M_j+\lambda)=\frac{\prod\limits_{j=1}^{n}\sigma_j}{\prod\limits_{j=1}^{n}\sigma_j-1}\Big(\sum_{k=m}^{n-1}\prod_{j=m+1}^{k}(M_j+\lambda)\prod_{s=k+2}^{n-1}M_s\Big)\lambda\,,^3 \tag{7}$$

where

$$M_j=\Big(\prod_{i=j}^{n-1}\Lambda_i\Big)^{-1}\ \ (j=\overline{m+1,n-1}),$$

does not have roots with zero real part, it is also sufficient that

$$\lim_{t\uparrow\omega}\frac{I_i(t)I_{i+1}'(t)}{I_i'(t)I_{i+1}(t)}=\Lambda_i\frac{\beta_{i+1}}{\beta_i}\ \ (i=\overline{1,n-1}),$$

and for each $i\in\{1,\dots,n\}$ *the following conditions are satisfied*

$$A_i^*\beta_i>0\ \ \textit{if}\ \ Y_i^0=\pm\infty,\quad A_i^*\beta_i<0\ \ \textit{if}\ \ Y_i^0=0,$$
$$\operatorname{sign}[\alpha_iA_i^*\beta_i]=\mu_i.$$

Moreover, components of each solution of that type admit the following asymptotic representation as $t\uparrow\omega$

$$\frac{y_i(t)}{\varphi_{i+1}(y_{i+1}(t))}=Q_i(t)[1+o(1)]\ \ (i=\overline{1,n-1}),$$
$$\frac{y_n(t)}{[\varphi_n(y_n(t))]^{\prod\limits_{i=1}^{n-1}\sigma_i}}=Q_n(t)[1+o(1)],$$

and there exists the whole k*-parametric family of these solutions if there are* k *positive roots among the solutions of the following algebraic equation*

$$\gamma_i=\begin{cases}\beta_iA_i^* & \textit{if}\ \ i\in\mathfrak{I}\setminus\{m+1,\dots,n-1\},\\ \beta_iA_i^*A_{i+1}^* & \textit{if}\ \ i\in\overline{\mathfrak{I}}\setminus\{m+1,\dots,n-1\},\\ A_n^*\Big(\prod\limits_{j=1}^{n-1}\sigma_j-1\Big)\operatorname{Re}\lambda_{i-m}^0 & \textit{if}\ \ i\in\{m+1,\dots,n\},\end{cases}$$

where λ_j^0 $(j=\overline{1,n-m})$ *are roots of the algebraic equation* (7) *(along with multiple ones).*

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