On Increasing the Order of Smallness of Fast Variables in Linear Differential Systems

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Let
$$G=\big\{t,\varepsilon:\ t\in[t_0,+\infty),\varepsilon\in(0,\varepsilon_0),\ \varepsilon_0\in\mathbf{R}^+\big\}.$$

Definition. We say that the function $f(t,\varepsilon)$ belongs to the class $S(m)$, $m\in\mathbf{N}\cup\{\mathbf{0}\}$, if:

1) $f:G\to\mathbf{C}$,

2) $f(t,\varepsilon)\in C^m(G)$ at t,

3) $d^kf(t,\varepsilon)/dt^k=\varepsilon^kf_k(t,\varepsilon)$ $(0\le k\le m)$,

$$\|f\|_{S(m)}\overset{def}{=}\sum_{k=0}^{m}\sup_G|f_k(t,\varepsilon)|<+\infty.$$

By slowly varying function we mean a function from the class $S(m)$.

Consider the system of differential equations
$$\frac{dx}{dt}=\Big(A_0(t,\varepsilon)+\sum_{s=1}^{r}A_s\big(t,\varepsilon,\theta(t,\varepsilon)\big)\big(\mu(\theta(t,\varepsilon))\big)^s\Big)x, \tag{1}$$
$x=\operatorname{colon}(x_1,\dots,x_n)$, $A_0(t,\varepsilon)$ – $(N\times N)$-matrix, whose elements belong to the class $S(m)$. The function $\theta(t,\varepsilon)$ has the form
$$\theta(t,\varepsilon)=\int_{t_0}^{t}\varphi(\tau,\varepsilon)\,d\tau, \tag{2}$$
$\varphi\in\mathbf{R}^+$, $\varphi(t,\varepsilon)\in S(m)$, $\inf\limits_G\varphi(t,\varepsilon)=\varphi_0>0$. The elements of matrices $A_s(t,\varepsilon,\theta)$ belong to the class $S(m)$ with respect to t, ε, are continuous and 2π-periodic with respect to $\theta\in[0,+\infty)$. The function $\mu(\theta)$ is continuous in $[0,+\infty)$.

With a small function $\mu(\theta)$ system (1) is close to the system with slowly varying coefficients
$$\frac{dx_0}{dt}=A_0(t,\varepsilon)x_0.$$
The terms depending on θ in system (1) has the order $O(\mu)$. We study the problem of reducing system (1) to the form where the terms depending on θ has the order $O(\mu^{r+1})$, or $O(\varepsilon)$. If a parameter ε is sufficiently small, then the transformed system will be closer to a system with slowly varying coefficients than to system (1).

Theorem. *Let system* (1) *satisfy the following conditions:*

1) *eigenvalues* $\lambda_j(t,\varepsilon)$ $(j=\overline{1,N})$ *of matrix* $A_0(t,\varepsilon)$ *are such that*
$$\lambda_j(t,\varepsilon)-\lambda_k(t,\varepsilon)=in_{jk}\varphi(t,\varepsilon),\ \ n_{jk}\in\mathbf{Z},$$
where the function $\varphi(t,\varepsilon)$ *are defined by condition* (2)*;*

2) *there exists a matrix* $L(t,\varepsilon)$*, the elements of which belong to the class* $S(m)$ *such that* $\inf\limits_G|\det L(t,\varepsilon)|>0$*, and*
$$L^{-1}(t,\varepsilon)A_0(t,\varepsilon)L(t,\varepsilon)=\Lambda(t,\varepsilon)=\operatorname{diag}\big[\lambda_1(t,\varepsilon),\dots,\lambda_N(t,\varepsilon)\big];$$

3) *the function* $\mu(\theta)$ *is such that*
$$\mu(\theta)\in\mathbf{R},\ \ \sup_{[\mathbf{0},+\infty)}\mu(\theta)\le\mu_{\mathbf{0}}<+\infty,\ \ \int\limits_{\mathbf{0}}^{+\infty}\mu^{\mathbf{k}}(\theta)\,\mathbf{d}\theta\le\mu_{\mathbf{0}}<+\infty\ \ (\mathbf{k}=\overline{\mathbf{1},\mathbf{r}}).$$

Then for sufficiently small values of μ_0 *there exists the transformation of the kind*
$$x=\Phi(t,\varepsilon,\theta(t,\varepsilon))y,$$
where the elements of the matrix $\Phi(t,\varepsilon,\theta(t,\varepsilon))$ *are bounded on* $G\times[t_0,+\infty)$*, that leads system* (1) *to the kind*
$$\frac{dy}{dx}=\big(\Lambda(t,\varepsilon)+\varepsilon V(t,\varepsilon,\theta)+W(t,\varepsilon,\theta)\big)y,\tag{3}$$
where the elements of the matrices $V(t,\varepsilon,\theta)$ *and* $W(t,\varepsilon,\theta)$ *are bounded on* $G\times[t_0,+\infty)$*, and the elements of the matrix* $W(t,\varepsilon,\theta)$ *has the order* μ_0^{r+1}*.*

Proof. We make in system (1) the substitution
$$x=L(t,\varepsilon)x^{(1)},$$
where $x^{(1)}$ – new unknown vector od dimension N. We obtain
$$\frac{dx^{(1)}}{dt}=\Big(\Lambda(t,\varepsilon)+\varepsilon H(t,\varepsilon)+\sum_{s=1}^{r}B_s(t,\varepsilon,\theta)(\mu(\theta))^s\Big)x^{(1)},\tag{4}$$
where
$$H(t,\varepsilon)=-\frac{1}{\varepsilon}L^{-1}(t,\varepsilon)\frac{dL(t,\varepsilon)}{dt}\,,\quad B_s(t,\varepsilon,\theta)=L^{-1}(t,\varepsilon)A_s(t,\varepsilon,\theta)L(t,\varepsilon).\tag{5}$$
The elements of the matrix $H(t,\varepsilon)$ belong to the class $S(m-1)$.

We seek the transformation, which leads system (4) to the kind (3), in the form
$$\frac{dx^{(1)}}{dt}=\Big(E+\sum_{s+1}^{r}Q_s(t,\varepsilon,\theta)\Big)y,\tag{6}$$
where the matrices $Q_s(t,\varepsilon,\theta)$ $(s=\overline{1,r})$ are defined from the next chain of the differential equations
$$\varphi(t,\varepsilon)\,\frac{\partial Q_1}{\partial\theta}=\Lambda(t,\varepsilon)Q_1-Q_1\Lambda(t,\varepsilon)+B_1(t,\varepsilon,\theta)\mu(\theta),\tag{7}$$
$$\varphi(t,\varepsilon)\,\frac{\partial Q_2}{\partial\theta}=\Lambda(t,\varepsilon)Q_2-Q_2\Lambda(t,\varepsilon)+B_2(t,\varepsilon,\theta)(\mu(\theta))^2+B_1(t,\varepsilon,\theta)Q_1t,\varepsilon,\theta)\mu(\theta),$$
. . . .
$$\varphi(t,\varepsilon)\,\frac{\partial Q_r}{\partial\theta}=\Lambda(t,\varepsilon)Q_r-Q_r\Lambda(t,\varepsilon)+B_r(t,\varepsilon,\theta)(\mu(\theta))^r+\sum_{s=1}^{r-1}B_s(t,\varepsilon,\theta)Q_{r-s}(t,\varepsilon,\theta)(\mu(\theta))^s.$$

The matrices $V(t,\varepsilon,\theta)$, $W(t,\varepsilon,\theta)$ are defined from the equations

$$\Big(E+\sum_{s=1}^{r}Q_s(t,\varepsilon,\theta)\Big)V=H(t,\varepsilon)\Big(E+\sum_{s=1}^{r}Q_s(t,\varepsilon,\theta)\Big)-\frac{1}{\varepsilon}\sum_{s=1}^{r}\frac{\partial Q_s(t,\varepsilon,\theta)}{dt}\,, \tag{8}$$

$$\Big(E+\sum_{s=1}^{r}Q_s(t,\varepsilon,\theta)\Big)W=\sum_{j=1}^{r}\sum_{s=j}^{r}B_s(t,\varepsilon,\theta)Q_{r+j-s}(t,\varepsilon,\theta)(\mu(\theta))^s. \tag{9}$$

Let

$$Q_s=(q_{jk}^{(s)})_{j,k=\overline{1,N}},\quad B_s=(b_{jk}^{(s)})_{j,k=\overline{1,N}},\quad s=\overline{1,r}.$$

Consider equation (7). By virtue condition 1) of the theorem equation (7) is equal to the set of scalar equations

$$\frac{\partial q_{jk}^{(1)}}{\partial\theta}=in_{jk}q_{jk}^{(1)}+\frac{1}{\varphi(t,\varepsilon)}\,\mu(\theta)b_{jk}^{(1)}(t,\varepsilon,\theta),\ \ j,k=\overline{1,N}. \tag{10}$$

For each of equations (10), we consider its solution

$$q_{jk}^{(1)}(t,\varepsilon,\theta)=\frac{1}{\varphi(t,\varepsilon)}\,e^{in_{jk}\theta}\int\limits_0^{\theta}\mu(\vartheta)b_{jk}^{(1)}(t,\varepsilon,\vartheta)e^{-in_{jk}\vartheta}d\vartheta,\ \ j,k=\overline{1,N}. \tag{11}$$

From the fact that elements of matrices $A_s(t,\varepsilon,\theta)$ in system (1) belong to the class $S(m)$ with respect to t,ε, and are continuous and 2π-periodic with respect to $\theta\in[0,+\infty)$, and from equality (5) it follows that the elements of the matrices $B_s(t,\varepsilon,\theta)$ also have similar properties. Hence

$$\sup_{G\times[0,+\infty)}|b_{jk}^{(1)}(t,\varepsilon,\theta)|=c_{jk}^{(1)}<+\infty,\ \ j,k=\overline{1,N}.$$

From (11) and condition 3) of the theorem we have

$$\sup_{G\times[0,+\infty)}\big|q_{jk}^{(1)}(t,\varepsilon,\theta)\big|\le\frac{1}{\varphi_0}\,\mu_0c_{jk}^{(1)},\ \ j,k=\overline{1,N}.$$

For $q_{jk}^{(r)}(t,\varepsilon,\theta)$ we define

$$q_{jk}^{(r)}(t,\varepsilon,\theta)=\frac{1}{\varphi(t,\varepsilon)}\,e^{in_{jk}\theta}\times\int\limits_0^{\theta}\Big((\mu(\vartheta))^rb_{jk}^{(r)}(t,\varepsilon,\vartheta)+\sum_{s=1}^{r-1}(\mu(\vartheta))^s\sum_{l=1}^{N}b_{jl}^{(s)}(t,\varepsilon,\vartheta)q_{lk}^{(r-k)}(t,\varepsilon,\vartheta)\Big)e^{-in_{jk}\vartheta}\,d\vartheta,\ \ j,k=\overline{1,N}.$$

All functions $q_{jk}^{(s)}(t,\varepsilon,\theta)$ $(j,k=\overline{1,N},\ s=\overline{1,r-1})$ are bounded at $t\in G\times[t_0,+\infty)$. All functions $b_{jk}^{(s)}(t,\varepsilon,\theta)$ $(j,k=\overline{1,N},\ s=\overline{1,r})$ are bounded also at $t\in G\times[t_0,+\infty)$. Hence, the condition 3) of the theorem guarantees existence of bounded solutions $q_{jk}^{(r)}(t,\varepsilon,\theta)$ $(j,k=\overline{1,N})$, and these solutions have the order μ_0^r. For the small μ_0 the same condition guarantees non-degeneracy of transformation (6). The matrix $V(t,\varepsilon,\theta)$ is uniquely defined from equation (8), and the matrix $W(t,\varepsilon,\theta)$ is uniquely defined from equation (9), and how easy it is to see that the order of the elements of the matrix $W(t,\varepsilon,\theta)$ are not less than μ_0^{r+1}. □