Definition and Properties of Perron Stability of Differential Systems

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1 The Perron stability definition

For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider the system

$$\dot{x} = f(t,x), \quad f(t,0) = 0, \;\; t \in \mathbb{R}^+ \equiv [0,\infty), \;\; x \in G, \tag{1.1}$$

with the right-hand side $f \in C^1(\mathbb{R}^+ \times G)$ admitting a zero solution. Let $\mathcal{S}_*(f)$ denote the set of all non-continuable nonzero solutions x of the system (1.1), then let $\mathcal{S}_\delta(f)$ and $\mathcal{S}^\delta(f)$ denote its subsets given by the initial conditions $|x(0)| < \delta$ and $|x(0)| = \delta$, respectively.

Definition 1.1. We say that a system (1.1) (more precisely, its zero solution, implied implicitly everywhere below) has the following *Perron features*:

(1) *Perron stability* if for any $\varepsilon > 0$ there is a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the requirement
$$\varliminf_{t\to\infty} |x(t)| < \varepsilon; \tag{1.2}$$

(2) *asymptotic Perron stability* if there is a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the requirement
$$\varliminf_{t\to\infty} |x(t)| = 0; \tag{1.3}$$

(3) *Perron instability* if there is no Perron stability, i.e. there is an $\varepsilon > 0$ such that for any $\delta > 0$ there is a solution $x \in \mathcal{S}_\delta(f)$ not satisfying the requirement (1.2) (in particular, not defined on the whole semi-axis $\mathbb{R}^+$);

(4) *complete Perron instability* if there are $\varepsilon, \delta > 0$ such that no solution $x \in \mathcal{S}_\delta(f)$ satisfies the requirement (1.2).

Remark 1.1. In Definition 1.1, each of the four Perron features:

(a) in a standard way (namely, with a simple shift of coordinates) extends from the *zero* solution to any other one, and not only to the points of rest of the system under study;

(b) is of a *local* character, i.e. it depends on the behavior of only those solutions that start near zero;

(c) characterizes the behavior of solutions starting near zero from the point of view of the possibility for them to *approach* the origin arbitrarily *late* or, conversely, *ultimately move away* from it.

The next two theorems describe some seemingly paradoxical situations.

Theorem 1.1. *There is a complete Perron unstable two-dimensional system* (1.1) *which has at least one solution $x \in \mathcal{S}_*(f)$ satisfying the requirement* (1.3) *and even the condition*

$$\lim_{t\to\infty} |x(t)| = 0. \tag{1.4}$$

Theorem 1.2. *There exists a Perron unstable two-dimensional autonomous system* (1.1) *such that for some $\delta > 0$ all solutions $x \in \mathcal{S}^\delta(f)$ satisfy the requirement* (1.4)*.*

2 Perron and Lyapunov stability joint properties

Definition 2.1 ([1, Ch. II, § 1])**.** Let us assign the *Lyapunov analogue* to each of the four Perron features above:

(a) *Lyapunov stability*, *instability* and *complete instability* are obtained by replacing the requirement (1.2) in the first, third and fourth paragraphs of the Definition 1.1 respectively by the following requirement

$$\sup_{t\in\mathbb{R}^+} |x(t)| < \varepsilon;$$

(b) *asymptotic Lyapunov stability* is obtained by replacing the requirement (1.3) in the second paragraph of the Definition 1.1 by the requirement (1.4), but with the Lyapunov stability.

Remark 2.1. For any system (1.1) the following *logical* statements are true:

(1) it is either Perron (Lyapunov) stable, or Perron (respectively, Lyapunov) unstable;

(2) if it is asymptotically Perron (Lyapunov) stable, then it is Perron (respectively, Lyapunov) stable;

(3) if it is completely Perron (Lyapunov) unstable, then it is Perron (respectively, Lyapunov) unstable;

(4) if it is Lyapunov stable (asymptotically), then it is Perron stable (respectively, asymptotically);

(5) if it is Perron unstable (completely), then it is Lyapunov unstable (respectively, completely).

Definition 2.2. We will call *strict* the following varieties of Perron (Lyapunov) features:

(a) asymptotic Perron (Lyapunov) stability;

(b) *non-asymptotic* Perron (Lyapunov) stability;

(c) complete Perron (Lyapunov) instability;

(d) *incomplete* Perron (Lyapunov) instability.

Consider a *linear* system of the form

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \tag{2.1}$$

defined by its continuous operator function $A : \mathbb{R}^+ \to \operatorname{End}\mathbb{R}^n$ (if it is bounded, we call the system *bounded* too). Denote by $\mathcal{S}^\delta_A$ the set of solutions x of the system (2.1) satisfying the initial condition $|x(0)| = \delta$.

All combinations of varieties of stability features from the Definition 2.2 which are logically admissible by the formulation of the previous remark turn out to be possible.

Theorem 2.1. *Any pair formed by any strict Perron and Lyapunov features and not conflicting with the statements of the Remark 2.1 is implemented in some at least two-dimensional bounded linear system* (2.1)*.*

A special role in the study on the stability of a linear (and not only) system is played by *characteristic exponents* of its solutions $x \in \mathcal{S}_*(f)$ – the *Lyapunov* ones [2, Ch. I] and, respectively, the *Perron* ones [3, § 2]

$$\lambda(x) \equiv \varlimsup_{t\to\infty} \frac{1}{t} \ln|x(t)|, \quad \pi(x) \equiv \varliminf_{t\to\infty} \frac{1}{t} \ln|x(t)|.$$

Theorem 2.2. *For each $n \in \mathbb{N}$ there is a complete Lyapunov unstable, but asymptotically (non-asymptotically) Perron stable n-dimensional bounded linear system* (2.1) *for which all Lyapunov exponents are positive and all Perron exponents are negative (respectively, equal to zero).*

From a practical point of view, the following two most *natural* situations seem to be particularly important:

(1) asymptotic Perron stability combined with Lyapunov stability;

(2) complete Perron (and, therefore, Lyapunov) instability.

3 The important special cases

If the system (1.1) is *one-dimensional*, then the verification of Perron features is somewhat simplified because of the possibility to order the solutions by increasing their initial values in the *numerical* phase straight line.

Theorem 3.1. *For a one-dimensional system* (1.1)*:*

(1) *Perron stability is equivalent to the fact that for any $\varepsilon > 0$ there exist two opposite-sign solutions $x \in \mathcal{S}_*(f)$ satisfying the requirement* (1.2)*;*

(2) *asymptotic Perron stability is equivalent to the existence of two opposite-sign solutions $x \in \mathcal{S}_*(f)$ satisfying the requirement* (1.3)*;*

(3) *complete Perron instability is equivalent to the existence of an $\varepsilon > 0$ such that for any $\delta > 0$ there are two opposite-sign solutions $x \in \mathcal{S}_\delta(f)$ that do not satisfy the requirement* (1.2)*.*

Remark 3.1. In the case of complete Perron instability, it is fundamentally excluded (due to the continuous dependence of the solutions on the initial values) the opportunity to find $\varepsilon, \delta > 0$, and $T \in \mathbb{R}$ such that *all at once* solutions $x \in \mathcal{S}_\delta(f)$ satisfy the requirement

$$\inf_{t\,T} |x(t)|\,\varepsilon. \tag{3.1}$$

Despite the Remark 3.1, in both one-dimensional and *autonomous* cases, the situation described in Theorem 1.1 is *impossible*, and the complete Perron instability still has a certain *uniformity*.

Theorem 3.2. *If a one-dimensional or autonomous system* (1.1) *is completely Perron unstable, then:*

(1) *for some $\varepsilon > 0$ no solution $x \in \mathcal{S}_*(f)$ satisfies the requirement* (1.2)*;*

(2) *for any $\delta > 0$ there exists an $\varepsilon > 0$ such that all solutions $x \in \mathcal{S}_*(f) \setminus \mathcal{S}_\delta(f)$ satisfy the requirement* (3.1) *already at $T = 0$.*

Each of the Perron features in the case of a *linear* system is completely determined by the properties of its solutions starting on some *sphere.*

Theorem 3.3. *The Perron stability of the linear system* (2.1) *is equivalent to fulfilling the requirement*

$$\sup_{x\in\mathcal{S}_A^1}\varliminf_{t\to\infty}|x(t)|<\infty,$$

and its asymptotic Perron stability or complete Perron instability is equivalent to the fact that any solution $x\in\mathcal{S}_A^1$ satisfies the requirement (1.3) *or, respectively, the requirement*

$$\lim_{t\to\infty}|x(t)|=\infty. \tag{3.2}$$

In the simplest case of a *linear autonomous* system the Perron and Lyapunov stability analysis lead to the *identical* result (unambiguously recognized by the real parts of the eigenvalues of the operator that defines the system and the orders of its Jordan cells corresponding to the purely imaginary ones [1, Ch. II, § 8]).

Theorem 3.4. *The linear autonomous system* (2.1) *is Perron stable (asymptotically stable, unstable, completely unstable) if and only if it is Lyapunov stable (respectively, asymptotically stable, unstable, completely unstable).*

The statement of Theorem 3.4 does *not extend* from autonomous linear systems to a slightly wider class of *regular* linear systems [1, Ch. III, § 11].

Theorem 3.5. *For each $n\in\mathbb{N}$ there exists a regular bounded linear system* (2.1) *that is asymptotically Perron stable, but completely Lyapunov unstable.*

In the case of a *linear* system, the fulfillment of the requirements (1.3) or (3.2) not for all its non-zero solutions, but only for those that constitute a *fundamental* solution system is not sufficient for Perron stability or, respectively, complete Perron instability.

Theorem 3.6. *For each $n>1$, there is an n-dimensional bounded linear system* (2.1) *with Perron instability (with incomplete instability) for which the Perron exponents of all solutions from some of its fundamental systems are negative (respectively, positive).*

However, in some (even non-linear) cases, the knowledge of the *set of exponents of all* solutions of the system starting close to zero gives full information about the Perron and Lyapunov features.

Theorem 3.7. *If for some $\delta>0$ the Perron (Lyapunov) exponents of all solutions $x\in\mathcal{S}_\delta(f)$ of the system* (1.1) *are negative, then the system is asymptotically Perron (respectively, Lyapunov) stable, and if they are positive, then it is completely unstable.*

4 The first-order stability

Let the *linear part* be distinguished in the right-hand side of the system (1.1), i.e. let it be represented as

$$\dot{x}=A(t)x+h(t,x)\equiv f(t,x),\ \ (t,x)\in\mathbb{R}^+\times G,\quad \sup_{t\in\mathbb{R}^+}|h(t,x)|=o(x),\ \ x\to0, \tag{4.1}$$

where $A(t)\equiv f'_x(t,0)$, $t\in\mathbb{R}^+$. Then for it the corresponding system (2.1) will be considered as the *first approximation* system.

Definition 4.1. We say that the first approximation system (2.1) *provides* a given Perron or Lyapunov feature if any system (4.1) with this first approximation has the given one.

The study of asymptotic stability by the first approximation, which is the essence of the first Lyapunov method, has been the subject of a huge number of works (see [3, § 11]). However, the study by the first approximation of stability or asymptotic stability, according to Perron or Lyapunov – all of them are possible only for *the same* systems.

Theorem 4.1. *If a linear approximation* (2.1) *provides at least one of the four features: Perron stability, Lyapunov stability, asymptotic Perron stability, or asymptotic Lyapunov stability – then it provides the other three of them.*

References

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