Investigation of Periodic Solutions of Autonomous System by Halving the Interval

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We study the T-periodic boundary value problem for the autonomous system of differential equations

$$u'(t) = f(u(t)), \ \ t \in [0,T]; \quad u(0) = u(T), \tag{1}$$

where T is the unknown period, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function defined on a closed bounded set (see (9)).

In [3] , we have suggested an approach for the investigation of general type of non-linear boundary value problem with the functional boundary conditions

$$u'(t) = f(t, u(t)), \ \ t \in [a,b], \ \ \Phi(u) = d, \tag{2}$$

where $\phi : C([a,b], \mathbb{R}^n)$ is a vector functional (possibly non-linear), which involves a kind of reduction to a parametrized family of problems with separated conditions

$$u'(t) = f(t, u(t)), \ \ t \in [a,b], \tag{3}$$

$$u(a) = z, \quad u(b) = \eta, \tag{4}$$

where $z := \operatorname{col}(z_1, \dots, z_n)$, $\eta := \operatorname{col}(\eta_1, \dots, \eta_n)$ are unknown parameters. The techniques of [3] are based on properties of the iteration sequence

$$\begin{aligned} u_m(t,z,\eta) &:= z + \int_a^t f(s, u_{m-1}(s,z,\eta))\,ds \\ &\quad - \frac{t-a}{b-a} \int_a^b f(s, u_{m-1}(s,z,\eta))\,ds + \frac{t-a}{b-a}\,[\eta - z], \ \ t \in [a,b], \ \ m = 1,2,\dots, \\ u_0(t,z,\eta) &:= z + \frac{t-a}{b-a}\,[\eta - z] \end{aligned} \tag{5}$$

and on the solution of the algebraic system

$$\Delta(z,\eta) := \eta - z - \int\limits_a^b f(s, u_m(s,z,\eta))\,ds. \tag{6}$$

Formulas (5), (6) are used to compute the corresponding functions explicitly for certain values of m, which, under additional conditions, allows one to prove the solvability of the problem and construct approximate solutions.

It is known, that the T-periodic solution $u^*(t)$ of autonomous system is not isolated in the extended phase space which means that every member of the one-parameter family of functions $t \to u^*(t+\varphi)$, $\varphi \in [0,T]$ is also a T-periodic solution. But, all these periodic solutions represent one and the same trajectory. In the autonomous T-periodic case (1) $z = \eta$ and the direct application of the successive approximation technique (5), (6) implies that

$$u_m(t,z,\eta) = z, \quad \Delta(z,\eta) = f(z) = 0.$$

Therefore, the successive approximations scheme determined by (5), (6) "detects" only constant stationary periodic solutions. In [1], it was considered the investigation of periodic solutions of autonomous systems by transforming them with special replacements into non-autonomous systems. Here we show that for the study of periodic solutions of autonomous systems it is advisable to use the technique of dividing a segment in half [2,4].

In view of the foregoing, without loss of generality, having replaced u^* by $u^*(\,\cdot\,+\varphi)$ with a suitable φ, we can assume in the subsequent consideration that a certain fixed, say jth component of the periodic function $u^*(\,\cdot\,+\varphi)$ takes extremal value over $[0,T]$ at the point $t=0$. So, we study the periodic solution of (1) for which

$$f_j\big(u_1(0), u_{21}(0), \dots, u_{n1}(0)\big) = 0. \tag{7}$$

Let us fix certain closed bounded sets $D_0, D_1 \subset \mathbb{R}^n$ and focus on the continuously differentiable T-periodic solutions u (T is unknown) of problem (1), (7) with values

$$u(0) \in D_0, \;\; u(T/2) \in D_1, \;\; u(T) \in D_0. \tag{8}$$

Based on the sets D_0 and D_1, we introduce the sets

$$D_{0,1} = (1-\theta)z + \theta\eta, \;\; z \in D_0, \;\; \eta \in D_1, \;\; \theta \in [0,1],$$

and its component-wise vector ρ neighborhood

$$D_\rho = O(D_{0,1}, \rho). \tag{9}$$

The problem is to find a continuously differentiable solution $u : [0,T] \to D_\rho$ to problem (1) for which inclusions (8) hold. We introduce the vectors of parameters

$$z = \operatorname{col}(z_1, z_2, \dots, z_n), \quad \eta = \operatorname{col}(\eta_1, \eta_2, \dots, \eta_n)$$

by formally putting

$$z := u(0), \quad \eta := u(T/2), \quad z := u(T). \tag{10}$$

Instead of (1) using a natural interval halving technique, we will consider on the intervals $t \in [0,T/2]$ and $[T/2,T]$, respectively, the following two "model-type" two-point problems with separated parametrised conditions

$$x'(t) = f(x(t)), \;\; t \in [0,T/2], \;\; x(0) = z, \;\; x(T/2) = \eta, \tag{11}$$

$$y'(t) = f(y(t)), \;\; t \in [T/2,T], \;\; y(T/2) = \eta, \;\; y(T) = z. \tag{12}$$

We suppose that

$$f \in \mathrm{Lip}(K, D_\rho) \ \text{ with the vector } \rho \text{ satisfying the inequality } \ \rho \geq \frac{T}{2}\,\delta_{D_\rho}(f), \tag{13}$$
$$r(Q) < 1, \ \text{ where } \ Q = \frac{3T}{20}\,K, \ \ \delta_{D_\rho}(f) = \frac{1}{2}\Big(\max_{x\in D_\rho} f(x) - \min_{x\in D_\rho} f(x)\Big).$$

To study the solutions of problems (11) and (12) let us introduce the following parametrised sequence of functions

$$\begin{aligned} x_m(t,z,\eta,T) &:= z + \int\limits_0^t f(x_{m-1}(s,z,\eta,T))\,ds \\ &\quad - \frac{2t-}{T}\int\limits_0^{T/2} f(x_{m-1}(s,z,\eta,T))\,ds + \frac{2t}{T}\,[\eta - z], \ \ t \in [0,T/2], \ \ m = 1,2,\dots, \\ x_0(t,z,\eta,T) &:= z + \frac{2t}{T}\,[\eta - z], \end{aligned} \tag{14}$$

and

$$\begin{aligned} y_m(t,z,\eta,T) &:= \eta + \int\limits_{T/2}^t f(y_{m-1}(s,z,\eta,T))\,ds - \frac{2(t-T/2)}{T}\int\limits_{T/2}^{T} f(y_{m-1}(s,z,\eta,T))\,ds \\ &\quad + \frac{2(t-T/2)}{T}\,[z-\eta], \ \ t \in [T/2,T], \ \ m = 1,2,\dots, \\ y_0(t,z,\eta,T) &:= \eta + \frac{2(t-T/2)}{T}\,[\eta - z]. \end{aligned} \tag{15}$$

Theorem 1. *Assume that for problem* (1) *conditions* (13) *are satisfied. Then for arbitrary* $(z,\eta) \in D_0 \times D_1$*:*

1. *All members of sequences* (14), (15) *are continuously differentiable functions on the intervals* $t \in [0,T/2]$ *and* $t \in [T/2,T]$ *satisfying conditions*
$$x_m(0,z,\eta) = z, \quad x_m(T/2,z,\eta) = \eta, \quad y_m(T/2,z,\eta) = \eta, \quad y_m(T,z,\eta) = z.$$
2. *Sequences* (14), (15) *in* $t \in [0,T/2]$ *and* $t \in [T/2,T]$*, respectively, converge uniformly as* $m \to \infty$ *to the limit functions*
$$x_\infty(t,z,\eta,T) = \lim_{m\to\infty} x_m(t,z,\eta,T), \quad y_\infty(t,z,\eta,T) = \lim_{m\to\infty} y_m(t,z,\eta,T).$$
3. *The limit functions are the unique continuously differentiable solution of the following additively perturbed equations for all* $(z,\eta) \in D_0 \times D_1$
$$x(t) := z + \int\limits_0^t f(x(s))\,ds - \frac{2t-}{T}\int\limits_0^{T/2} f(x(s))\,ds + \frac{2t}{T}\,[\eta - z], \ t \in [0,T/2],$$
$$y(t) := \eta + \int\limits_{T/2}^t f(y(s))\,ds - \frac{2(t-T/2)}{T}\int\limits_{T/2}^{T} f(y(s))\,ds + \frac{2(t-T/2)}{T}\,[z-\eta], \ \ t \in [T/2,T].$$

Theorem 2. *Let the conditions of Theorem 1 hold. Then the function*

$$u_\infty(t)=\begin{cases} x_\infty(t,z,\eta), & t\in[0,T/2]\\ x_\infty(t,z,\eta), & t\in[T/2,T]\end{cases}$$

is a continuously differentiable T-periodic solution of (1) *if and only if the triplet (z,η,T) satisfies the system of $2n+1$ algebraic or transcendental determining equations*

$$\Delta(z,\eta,T)=\eta-z-\int\limits_0^{T/2} f(x_\infty(s,z,\eta,T))\,ds=0, \tag{16}$$

$$H(z,\eta,T)=\eta-z-\int\limits_{T/2}^{T} f(y_\infty(s,z,\eta,T))\,ds=0,$$

$$f_j(z_1,z_2,\dots,z_n)=0.$$

Note that the solvability of (1) *can be established by studying the approximate determining system, when in* (16) *instead of ∞ stands m.*

Let us apply the approach described above to the system

$$\begin{aligned} \frac{du_1}{dt}&=u_2,\\ \frac{du_2}{dt}&=-4u_1+u_1^2+\frac{u_2^2}{16}-\frac{1}{64}\,. \end{aligned} \tag{17}$$

The domains D_0, D_1, vector ρ can be choosen to satisfy the conditions of Theorem 1. Applying Maple (14), we carried out the calculations. Note that as a zeroth approximation in formulas (14), (15), one can choose any function with values in domain D_ρ.

Introduce the following parameters $z=\mathrm{col}(z_1,z_2),\eta=\mathrm{col}(\eta_1,\eta_2)$. If in (7) $j=1$, then from (17) it follows that $z_2=0$. The system (17) has two stationary constant solutions

$$z_1=-0.9765029026\cdot10^{-3},\ \ z_2=0\ \text{ and }\ z_1=16.00097650,\ z_2=0.$$

The exact $\frac{\pi}{2}$-periodic solution of system (17) is $u_1(t)=\frac{1}{8}\,\cos(4t)$, $u_2(t)=-\frac{1}{2}\,\sin(4t)$. For a different number of approximations m, we obtain from (14), (15) and from the approximate determining system (16) the following numerical values for the introduced parameters which are presented in Table 1.

Table 1.

m	z_1	η_1	η_2	T
0	0.09964844522	−0.1003515548	$1.079348881\cdot10^{-12}$	1.570796327
1	0.09965288938	−0.1003558995	$3.849442526\cdot10^{-12}$	1.570796327
3	0.0996603478	−0.1003631726	$-1.637826662\cdot10^{-12}$	1.570796327
Exact	0.125	−0.1250000000	0	1.570796327

Note that the second equilibrium point and the $\frac{\pi}{2}$-periodic solution are unstable.

On Figure 1, we have the graphs of the exact solution (solid line) and its third approximation $(\times)$ for the first and second components on the intervals $t\in[0,T]$.

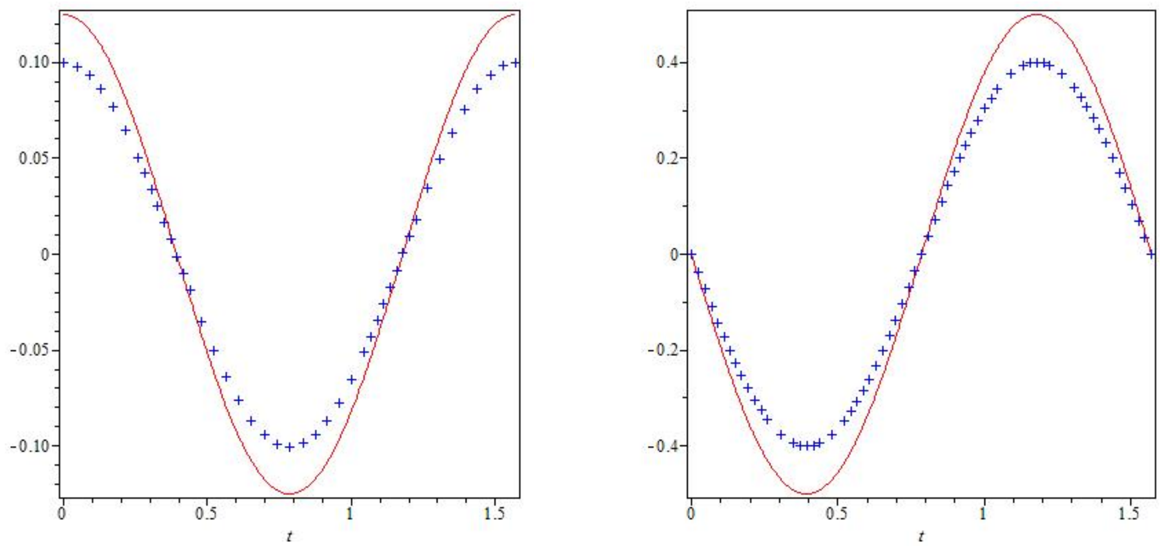

Figure 1.

References

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