Antiperiodic Problem with Barriers

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1 Introduction

Some real world models are described by means of impulse control of nonlinear BVPs, where time instants of impulse actions depend on intersection points of solutions with given barriers. For $i = 1, \dots, m$, and $[a, b] \subset \mathbb{R}$, continuous functions $\gamma_i : \mathbb{R} \to [a, b]$ determine barriers $\Gamma_i = \{(t, z) : t = \gamma_i(z),\ z \in \mathbb{R}\}$. A solution (x, y) of a planar BVP on $[a, b]$ is searched such that the graph of its first component $x(t)$ has exactly one intersection point with each barrier, i.e. for each $i \in \{1, \dots, m\}$ there exists a unique root $t = t_{ix} \in [a, b]$ of the equation $t = \gamma_i(x(t))$. The second component $y(t)$ of the solution has impulses (jumps) at the points $t_{1x}, \dots, t_{mx}$. Since a size of jumps and especially the points $t_{1x}, \dots, t_{mx}$ depend on x, impulses are called *state-dependent*.

More precisely, for $T > 0$ and given continuous functions $\gamma_1, \dots, \gamma_m$, we prove the existence of a T-antiperiodic solution (x, y) of the van der Pol equation with a positive parameter μ and a Lebesgue integrable T-antiperiodic function f

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \ \text{ for a.e. } \ t \in [0, T], \ \ t \notin \{t_{1x}, \dots, t_{mx}\}, \tag{1.1}$$

where y has impulses at the points $t_{1x}, \dots, t_{mx} \in (0, T)$ determined by the barriers $\Gamma_1, \dots, \Gamma_m$ through the equalities

$$t_{ix} = \gamma_i(x(t_{ix})), \ \ i = 1, \dots, m, \tag{1.2}$$

and y is continuous anywhere else in $[0, T]$. The impulse conditions have the form

$$y(t+) - y(t-) = \mathcal{J}_i(x), \ \ t = t_{ix}, \ \ i = 1, \dots, m, \tag{1.3}$$

where $\mathcal{J}_i$ are continuous bounded functionals defining a size of jumps.

Previous results in the literature for this antiperiodic problem assume that impulse points are values of given continuous functionals, see [1, 3]. Such formulation is certain handicap for applications to real world problems where impulse instants depend on barriers. We have found conditions which enable to reach such functionals from given barriers. Consequently the existence results from [2] for impulsive antiperiodic problem to the van der Pol equation formulated in terms of barriers are obtained.

Notations

- T-antiperiodic function x (satisfying (1.1), (1.2), (1.3)) will be found in the set of $2T$-periodic real-valued functions. To do it functional sets defined below are used.

- L^1 consists of $2T$-periodic Lebesgue integrable functions on $[0, 2T]$ with the norm $\|x\|_{\mathrm{L}^1} := \frac{1}{2T} \int\limits_0^{2T} |x(t)| \, \mathrm{d}t$,

- BV consists of $2T$-periodic functions of bounded variation on $[0,2T]$,
- $\operatorname{var}(x)$ for $x \in \mathrm{BV}$ is the total variation of x on $[0,2T]$,
- $\|x\|_\infty := \sup\{|x(t)| : t \in [0,2T]\}$ for $x \in \mathrm{BV}$,
- NBV consists of normalized functions $x \in \mathrm{BV}$ in the sense that $x(t) = \frac{1}{2}\,(x(t+) + x(t-))$,
- $\overline{x} := \frac{1}{2T} \int\limits_0^{2T} x(t)\,\mathrm{d}t = 0$ is the mean value of $x \in \mathrm{BV}$,
- $\widetilde{\mathrm{NBV}}$ consists from functions $x \in \mathrm{NBV}$ with $\overline{x} = 0$; $\widetilde{\mathrm{NBV}}$ with the norm $\operatorname{var}(x)$ is the Banach space,
- $\mathrm{AC}(J)$ consists of $2T$-periodic absolutely continuous functions on $J \subset [0,2T]$ and if $J = [0,2T]$ we write AC,
- $\widetilde{\mathrm{AC}} := \mathrm{AC} \cap \widetilde{\mathrm{NBV}}$.
- A couple $(x,y) \in \widetilde{\mathrm{AC}} \times \widetilde{\mathrm{NBV}}$ satisfying (1.1), (1.2), (1.3) is a $2T$-periodic *solution* of problem (1.1)–(1.3). If in addition
$$x(0) = -x(T), \quad y(0) = -y(T), \tag{1.4}$$
then (x,y) is a T-*antiperiodic solution* of problem (1.1)–(1.3).

2 Main result

The main existence result from [2] is contained in the next theorem.

Theorem 2.1 (Main result)**.** *Let $T \in (0,\sqrt{3})$, $K, L \in (0,\infty)$, let $\mathcal{J}_i$, $i = 1,\dots,m$, be continuous bounded functionals on $\widetilde{\mathrm{NBV}}$, and let $f \in \mathrm{L}^1$ be T-antiperiodic, i.e. $f(t+T) = -f(t)$ for a.e. $t \in \mathbb{R}$. Assume that there exist $a, b \in (0,T)$ such that functions $\gamma_1,\dots,\gamma_m$ satisfy*
$$0 < a \le \gamma_1(z) < \gamma_2(z) < \dots < \gamma_m(z) \le b < T, \quad z \in [-K,K]. \tag{2.1}$$
Further, assume that $L_i \in (0,1/L)$, $i = 1,\dots,m$, are such that
$$|\gamma_i(z_1) - \gamma_i(z_2)| \le L_i|z_1 - z_2|, \quad z_1, z_2 \in [-K,K], \quad i = 1,\dots,m. \tag{2.2}$$

Then there exists $\mu_0 > 0$ such that for each $\mu \in (0,\mu_0]$ problem (1.1)–(1.3) *has a T-antiperiodic solution (x,y), where y has m jumps at the points $t_{1x},\dots,t_{mx} \in [a,b]$ and y is continuous anywhere else in $[0,T]$. Moreover, the estimate*
$$|x(t)| \le \operatorname{var}(x) \le K, \quad |y(t)| \le L, \quad t \in [0,T], \tag{2.3}$$
is valid.

We can find the optimal (maximal) value μ_0 as follows. Since $\mathcal{J}_i$ are bounded, it holds
$$\mathcal{J}_i : \widetilde{\mathrm{NBV}} \to [-a_i, a_i], \quad i = 1,\dots,m,$$
for some $a_i \in (0,\infty)$. Denote
$$c_1 := T\|f\|_{\mathrm{L}^1} + \sum_{i=1}^m a_i, \tag{2.4}$$

and define a function φ by

$$\varphi(\mu) := \frac{1-\mu T-\frac{T^2}{3}}{3}\sqrt{\frac{1-\mu T-\frac{T^2}{3}}{\mu T}}\,, \ \ \mu \in \Big(0, \frac{1}{T}-\frac{T}{3}\Big]. \tag{2.5}$$

Then, according to the proof of Theorem 2.1, $\mu_0 = \varphi^{-1}(Tc_1) \in (0, \frac{1}{T}-\frac{T}{3})$.

Auxiliary results

Denote

$$(x*y)(t) := \frac{1}{2T}\int_0^{2T} x(t-s)y(s)\,\mathrm{d}s, \ \ t\in[0,2T] \ \text{ for } \ x,y\in \mathrm{L}^1,$$

and remind the inequalities

$$\operatorname{var}(x*y) \le \operatorname{var}(x)\|y\|_\infty, \ \ x,y\in \mathrm{NBV}, \tag{2.6}$$
$$\operatorname{var}(x*f) \le \operatorname{var}(x)\|f\|_{\mathrm{L}^1}, \ \ x\in \mathrm{NBV}, \ f\in \mathrm{L}^1, \tag{2.7}$$
$$\|x\|_{\mathrm{L}^1} \le \|x\|_\infty \le \operatorname{var}(x), \ \ x\in \widetilde{\mathrm{NBV}}. \tag{2.8}$$

Further, using the function

$$E_1(t) = \begin{cases} T-t & \text{for } \ t\in(0,2T),\\ 0 & \text{for } \ t=0, \end{cases}$$

which fulfils

$$\operatorname{var}(E_1) = 4T, \ \ \|E_1\|_\infty = T, \tag{2.9}$$

we introduce antiderivative operators I and I^2 by

$$Iu := E_1 * u \in \widetilde{\mathrm{AC}}, \ \ I^2u := I(Iu) \in \widetilde{\mathrm{AC}}, \ \ u\in \mathrm{L}^1. \tag{2.10}$$

For $\tau\in\mathbb{R}$ we define a distribution ε_τ by the Fourirer series

$$\varepsilon_\tau := \sum_{n\in\mathbb{Z}} \big(1-(-1)^n\big)\mathrm{e}^{\frac{in\pi}{T}(t-\tau)}, \ \ t\in\mathbb{R}. \tag{2.11}$$

Then it holds

$$I\varepsilon_\tau \in \widetilde{\mathrm{NBV}}, \ \ I^2\varepsilon_\tau \in \widetilde{\mathrm{AC}}, \ \ \|I\varepsilon_\tau\|_\infty = T. \tag{2.12}$$

See [3] for more details. Using this we investigated in [3] the van del Pol equation

$$x'(t) = y(t), \quad y'(t) = \mu\Big(x(t) - \frac{x^3(t)}{3}\Big)' - x(t) + f(t) \ \text{ for a.e. } \ t\in\mathbb{R}, \tag{2.13}$$

with a positive parameter μ, a Lebesgue integrable T-antiperiodic function f, and with the state-dependent impulse conditions

$$\lim_{t\to\tau_i(x)+} y(t) - \lim_{t\to\tau_i(x)-} y(t) = \mathcal{J}_i(x), \ \ i=1,\dots,m, \tag{2.14}$$

where $\mathcal{J}_i$ and also τ_i, $i=1,\dots,m$, are given continuous and bounded real-valued functionals on $\widetilde{\mathrm{NBV}}$. For such setting we proved the existence result contained in Theorem 2.2.

Theorem 2.2 ([3, Theorem 1.1])**.** *Assume that $T\in(0,\sqrt{3})$, and the functionals $\tau_1,\dots,\tau_m$ have values in $(0,T)$. Further, let*

$$i\neq j \implies \tau_i(x)\neq\tau_j(x), \ \ x\in\widetilde{\mathrm{AC}}, \ \ i,j=1,\dots,m. \tag{2.15}$$

Then there exists $\mu_0>0$ such that for each $\mu\in(0,\mu_0]$ the problem (2.13), (2.14) has a T-antiperiodic solution (x,y).

3 Existence of continuous functionals

If we study problem (1.1)–(1.3) which is formulated by means of barriers, then a number of impulse points for some solution (x, y) of (1.1) is equal to a number of values of t satisfying the equations $t - \gamma_i(x(t)) = 0$, $i = 1, \dots, m$. In general, for any (x, y) satisfying (1.1), such equations need not be solvable, or they can have finite or infinite number of roots. In Theorem 2.1, we present conditions imposed on barriers which yield unique solvability of these equations provided x belongs to some suitable set Ω_{KL} (see (3.1)). This yields functionals continuous on Ω_{KL}. We prove it in the next lemmas.

In particular, for positive numbers K and L, define a set Ω_{KL}

$$\Omega_{KL} := \big\{x \in \widetilde{\mathrm{AC}} : \ \mathrm{var}(x) \le K, \ |x'(t)| \le L \text{ for a.e. } t \in [0, 2T], \ x \text{ is } T\text{-antiperiodic}\big\}. \tag{3.1}$$

Lemma 3.1. *The set Ω_{KL} is nonempty, bounded, convex and closed in $\widetilde{\mathrm{NBV}}$.*

Lemma 3.2. *Let $K, L \in (0, \infty)$. Assume that there exist $a, b \in (0, T)$ and $L_i \in (0, 1/L)$, $i = 1, \dots, m$, such that (2.1) and (2.2) are fulfilled. Then for each $x \in \Omega_{KL}$ and $i \in \{1, \dots, m\}$ the equation*

$$t = \gamma_i(x(t)) \tag{3.2}$$

has a unique solution $t_{ix} \in [a, b]$.

Lemma 3.3. *Let the assumptions of Lemma 3.2 be fulfilled. Then for $i \in \{1, \dots, m\}$, the functional*

$$\tau_i : \Omega_{KL} \to [a, b], \ \ \tau_i(x) = t_{ix}, \tag{3.3}$$

where t_{ix} is a solution of (3.2), is continuous.

Having continuous functionals $\tau_1, \dots, \tau_m$ from Lemma 3.3, we can argue similarly as in [3] in the proof of Theorem 2.2 and prove Theorem 2.1.

References

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