A Class of Continuous-Discrete Functional Differential Equations with the Cauchy Operator Constructed Explicitly

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1 Introduction

Here we follow the previous works [2–4] and consider the linear continuous-discrete functional differential system

$$\delta y = \mathcal{T} y + r, \tag{1.1}$$

where $y = col(x, z), r = col(f, g), x : [0, T] \to R^n$, $z : \{0, t_1, \dots, t_\mu\} \to R^\nu$, $\delta y = col(\dot{x}, z)$, $\mathcal{T} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix}$, and $\mathcal{T}_{11} : AC^n \to L^n$, $\mathcal{T}_{12} : FD^\nu \to L^n$, $\mathcal{T}_{21} : AC^n \to FD^\nu$, $\mathcal{T}_{22} : FD^\nu \to FD^\nu$ are linear Volterra operators. Here L^n is the space of summable functions $f : [0, T] \to R^n$, AC^n is the space of absolutely continuous functions $x : [0, T] \to R^n$, the space FD^ν is defined by the given set $J = \{0, t_1, \dots, t_\mu\}, 0 = t_0 < t_1 < \dots < t_\mu = T$, as the space of functions $z : J \to R^\nu$. The spaces L^n, AC^n and FD^ν are assumed to be equipped with natural norms.

It should be noted that the system (1.1) can be considered as a concrete realization of the so-called Abstract Functional Differential Equation, the theory of which is thoroughly treated in [1]. The systems of the kind (1.1) arise in particular as dynamic models in Mathematical Economics and cover many kinds of systems with aftereffect. Representation of solutions to some classes of dynamic models close to (1.1) and discussion of actual applied problems can be found in [10]. The questions of stability to functional differential systems with continuous and discrete times are studied in [13].

The central point of the consideration is the representation of solutions to (1.1). The structure and some principal properties of the Cauchy operator are described in [8] with the use of the general representation to the operators $\mathcal{T}_{ij}$, $i, j = 1, 2$. The main aim of this paper is to give an explicit representation for the components of the Cauchy operator in a special case.

2 The Cauchy operator

Let V be the integration operator: $(Vu)(t) = \int\limits_0^t u(s)\,ds$, and $K = \mathcal{T}_{11} V$ be an integral operator with the kernel $K(t, s) = (k_{ij}(t, s))$ that satisfies the condition $\mathcal{K}$: for all the elements k_{ij}, there exists a common summable majorant $\kappa(\cdot)$, $|k_{ij}(t, s)| \le \kappa(t), t \in [0, T]$.

Let us recall some general results [2] for the case that the condition $\mathcal{K}$ is fulfilled.

The general solution of (1.1) has the representation

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{X} \begin{pmatrix} x(0) \\ z(0) \end{pmatrix} + \mathcal{C} \begin{pmatrix} ccf \\ g \end{pmatrix},$$

where $\mathcal{X}=\begin{pmatrix}\mathcal{X}_{11} & \mathcal{X}_{12}\\ \mathcal{X}_{21} & \mathcal{X}_{22}\end{pmatrix}$ is the fundamental operator (fundamental matrix), $\mathcal{C}=\begin{pmatrix}\mathcal{C}_{11} & \mathcal{C}_{12}\\ \mathcal{C}_{21} & \mathcal{C}_{22}\end{pmatrix}$ is the Cauchy operator.

Denote by C_1 and $X(t)$ the Cauchy operator and the fundamental matrix to the equations $\dot{x}=\mathcal{T}_{11}x$, and denote by C_2 and $Z(t_i)$ the Cauchy operator and the fundamental matrix to the equation $z=\mathcal{T}_{22}z$.

Define the operators $\mathcal{H}_{ij}$, $i,j=1,2$ by the equalities

$$\mathcal{H}_{11}=(I-C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1},\quad \mathcal{H}_{12}=-(I-C_1\mathcal{T}_{12}C_2\mathcal{T}_{12})^{-1}C_1\mathcal{T}_{21},$$
$$\mathcal{H}_{21}=C_2\mathcal{T}_{21}(I-C_1\mathcal{T}_{12}C_2\mathcal{T}_{21})^{-1},\quad \mathcal{H}_{22}=(I-C_2\mathcal{T}_{21}C_1\mathcal{T}_{12})^{-1},$$

where I is the identity operator.

Theorem 2.1 ([9])**.** *The Cauchy operator $\mathcal{C}=(\mathcal{C}_{ij})$ of* (1.1) *is defined by the equalities*

$$\mathcal{C}_{ij}=\mathcal{H}_{ij}C_j,\ \ i,j=1,2.$$

It should be noted that C_2 can be constructed in the explicit form. From Theorem 2.1 it follows that the component C_1 is of principal interest and requires the development of efficient algorithms to approximate construction of it. Some of those are described in [7].

In what follows we shall construct the Cauchy operator for the following continuous-discrete functional differential system

$$\dot{x}(t)=\sum_{i:\,t_i<t}A_i(t)x(t_i)+\sum_{i:\,t_i<t}B_i(t)z(t_i)+f(t),\ \ t\in[0,T], \tag{2.1}$$
$$z(t_i)=\sum_{j<i}D_jx(t_j)+\sum_{j<i}H_jz(t_j)+g(t_i),\ \ i=1,\dots,\mu \tag{2.2}$$

with summable $(n\times n)$-matrices $A_i(t)$, $(n\times\nu)$-matrices $B_i(t)$ and constant $(\nu\times n)$-matrices D_j, $(\nu\times\nu)$-matrices H_j.

Let us define the operator $\varTheta: AC^n\to L^n$ by the equality

$$(\varTheta\, x)(t)=\sum_{i:\,t_i<t}A_i(t)x(t_i)+\sum_{i:\,t_i<t}B_i(t)\Big[\sum_{j=1}^{i}C_2(i,j)\sum_{k<j}D_k\,x(t_k)\Big].$$

After some transformations this operator can be represented in the form

$$(\varTheta\, x)(t)=\sum_{i:\,t_i<t}\mathcal{A}_i(t)x(t_i), \tag{2.3}$$

where the matrices $\mathcal{A}_i(t)$ are calculated by $A_i(t)$, $B_i(t)$, $C_2(i,j)$, D_i.

Denote by $C(t,s)$ the Cauchy matrix [5] to the equation $\dot{x}=\varTheta x$.

As is shown in [7, Theorem 1, Remark 2], $C(t,s)$ can be constructed explicitly. Let us recall the main relationships from [7]. Let $\eta_i(t)$, $i=1,\dots,\mu-1$, be the characteristic function of the set $[t_{i-1},t_i)$, and $\eta_\mu(t)$ denotes the characteristic function of the segment $[t_{\mu-1},t_\mu]$. Define the kernel of the integral operator $(Kz)(t)=\int\limits_0^t K(t,s)z(s)\,ds$ by the equality

$$K(t,s)=\sum_{i=1}^{\mu}\sum_{j=1}^{i}\eta_i(t)P_i(t)Q_{ij}(s)\eta_j(s), \tag{2.4}$$

where $P_i(t)$ and $Q_{ij}(s)$ are $(n \times n)$-matrices, $P_1(t) = 0$, $Q_{ij}(s) = 0$, $j \geq i$, elements of P_i are summable on $[0,T]$, elements of Q_{ij} are measurable and essentially bounded on $[0,T]$. Next define the matrices B_{ki} by the equalities

$$B_{ki} = \int_0^T \sum_{j=1}^k Q_{kj}(t)\eta_j(t)\eta_i(t)P_i(t)\,dt.$$

Notice that by definition the block matrix $G = \{G_{ki}\}_{k,i=1,\dots,\mu}$, $G_{kk} = E_n$, $k = 1,\dots,\mu$, where E_n is the identity $(n \times n)$-matrix, $G_{ki} = -B_{ki}$, is a lower triangle matrix with E_n as the diagonal blocks. Finally denote by F_{ki} the block elements of the inverse G^{-1}. By Theorem 1 of [7] we have the explicit representation of the resolvent kernel $R(t,s)$ for the kernel $K(t,s)$ defined by (2.4):

$$R(t,s) = \sum_{i=1}^{\mu}\sum_{k=1}^{\mu}\sum_{j=1}^{k} \eta_i(t)P_i(t)F_{ik}Q_{kj}(s)\eta_j(s),$$

and

$$C(t,s) = E_n + \int_s^t R(\tau,s)\,d\tau.$$

It remains to note that, for the operator Θ (2.3), we have

$$(\Theta Vu)(t) = \int_0^t \sum_{i:\, t_i<t} \mathcal{A}_i(t)\eta_i(s)u(s)\,ds,$$

and this is the integral operator with the kernel of the kind (2.4). Now we are ready to give the representations of the fundamental matrix $\mathcal{X}$ and the Cauchy operator $\mathcal{C}$ for the system (2.1), (2.2) in terms of $X(t)$, $Z(t_i)$, $C(t,s)$ and $C_2(i,j)$.

Theorem 2.2. *The representation of the components to the fundamental matrix and the Cauchy operator of* (2.1), (2.2) *is defined by the equalities*

$$\mathcal{X}_{11}(t) = X(t), \quad \mathcal{X}_{12}(t) = \int_0^t C(t,s)\Big[\sum_{i:\, t_i<t} B_i(t)Z(t_i)\Big]\,ds,$$

$$\mathcal{X}_{21}(t_i) = \sum_{j=1}^{i} C_2(i,j)\Big[\sum_{k<j} D_k\mathcal{X}_{11}(t_k)\Big], \quad \mathcal{X}_{22}(t_i) = Z(t_i) + \sum_{j=1}^{i} C_2(i,j)\Big[\sum_{k<j} D_k\mathcal{X}_{11}(t_k)\Big],$$

$$(\mathcal{C}_{11}f)(t) = \int_0^t C(t,s)f(s)\,ds, \quad (\mathcal{C}_{12}g)(t) = \int_0^t C(t,s)\sum_{i:\, t_i<s} B_i(s)\Big[\sum_{j=1}^{i} C_2(i,j)g(t_j)\Big]\,ds,$$

$$(\mathcal{C}_{21}f)(t_i) = \sum_{j=1}^{i} C_2(i,j)\Big[\sum_{k<j} D_k(\mathcal{C}_{11}f)(t_k)\Big], \quad (\mathcal{C}_{22}g)(t_i) = \sum_{j=1}^{i} C_2(i,j)\Big[\sum_{k<j} D_k(\mathcal{C}_{12}g)(t_k) + g(t_j)\Big].$$

The systems (2.1), (2.2) are actively studied as models of some dynamic economic processes [12]. Furthermore, they can be used as approximations of more general systems (1.1) which opens the way to obtaining external estimates of attainability sets for control problems [6, 11].

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