Characterizing the Formation of Singularities in a Superlinear Indefinite Mean Curvature Problem

Julian López-Gómez

Departamento de Matemática Aplicada, Universidad Complutense de Madrid, Madrid, Spain
E-mail: julian@mat.ucm.es

Pierpaolo Omari

Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, Trieste, Italy
E-mail: omari@units.it

In this contribution, based on the very recent paper [21], we analyze the quasilinear indefinite Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = \lambda a(x) f(u) \quad \text{in } \ (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{1}$$

Here, $\lambda \in \mathbb{R}$ is regarded as a parameter and

(a_1) the function $a \in L^\infty(0,1)$ satisfies, for some $z \in (0,1)$, $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$, as well as $\int\limits_0^1 a(x)\,dx < 0$;

(f_1) the function $f \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1[0,+\infty)$ satisfies $f(s) > 0$ and $f'(s) \geq 0$ for all $s > 0$, and there exist four constants, $h > 0$, $k > 0$, $q > 1$ and $p \geq 2$, such that

$$\lim_{s \to +\infty} \frac{f(s)}{s^{q-1}} = qh, \quad \lim_{s \to 0^+} \frac{f(s)}{s^{p-1}} = pk.$$

Condition (f_1) implies that the potential F of f, defined by $F(s) = \int\limits_0^s f(t)\,dt$, satisfies

$$\lim_{s \to +\infty} \frac{F(s)}{s^q} = h, \quad \lim_{s \to 0^+} \frac{F(s)}{s^p} = k$$

and, thus, F must be superlinear at $+\infty$ and either quadratic or superquadratic at 0. We also introduce the following condition on the weight function a at the nodal point z, which is going to play a pivotal role in the mathematical analysis carried out in [21]

(a_2) $\quad \Big(\int\limits_x^z a(t)\,dt\Big)^{-\frac{1}{2}} \in L^1(0,z)$ and $\Big(\int\limits_x^z a(t)\,dt\Big)^{-\frac{1}{2}} \in L^1(z,1)$.

We use the following notions of a solution.

- A couple (λ, u) is said to be a *regular solution* of (1) if $u \in W^{2,1}(0,1)$ and it satisfies the differential equation a.e. in $(0,1)$, as well as the boundary conditions.

- A couple (λ, u) is said to be a *bounded variation solution* of (1) if $u \in BV(0,1)$ and it satisfies

$$\int_0^1 \frac{D^a u\, D^a \phi}{\sqrt{1+|D^a u|^2}}\,dx + \int_0^1 \frac{D^s u}{|D^s u|}\,D^s \phi = \int_0^1 \lambda a f(u)\phi\,dx$$

for all $\phi \in BV(0,1)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s u|$ (cf. [2]).

- A couple (λ, u) is said to be a *singular solution* of (1) whenever it is a non-regular bounded variation solution; that is, $u \in BV(0,1) \setminus W^{2,1}(0,1)$.

- When the couple (λ, u) solves (1) in any of the previous senses, it is said that (λ, u) is a *positive solution* if, in addition,

$$\lambda > 0, \quad \operatorname{ess\,inf} u > 0.$$

As usual, for any function $v \in BV(0,1)$,

$$Dv = D^a v\,dx + D^s v$$

stands for the Lebesgue decomposition of the Radon measure Dv and $\frac{D^s v}{|D^s v|}$ denotes the density function of the measure $D^s v$ with respect to its total variation $|D^s v|$ (see [1]). By [23, Prop. 3.6], any positive singular solution, (λ, u), of (1) actually satisfies

$$\begin{aligned} u\big|_{[0,z)} &\in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z) \ \text{ and is concave,}\\ u\big|_{(z,1]} &\in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1) \ \text{ and is convex;} \end{aligned} \tag{2}$$

moreover, $u'(x) < 0$ for every $x \in (0,1) \setminus \{z\}$, $u'(0) = u'(1) = 0$ and

$$u'(z^-) = u'(z^+) = -\infty,$$

where $u'(z^-)$ and $u'(z^+)$ are the left and the right Dini derivatives of u at z. The same argument used in [23, Lem. 2.1] shows that $\lambda > 0$ is necessary for the existence of positive non-constant, either regular or singular, solutions.

Problem (1) is a one-dimensional prototype model of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = g(x,u) & \text{in } \Omega,\\ -\dfrac{\nabla u\cdot\nu}{\sqrt{1+|\nabla u|^2}} = \sigma & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. Problem (3) plays a central role in the mathematical analysis of a number of geometrical and physical issues, such as prescribed mean curvature problems for cartesian surfaces in the Euclidean space [3, 9, 12–15, 19, 25, 26], capillarity phenomena for incompressible fluids [6, 10, 11, 16, 17], and reaction-diffusion processes where the flux features saturation at high regimes [5, 18, 24].

The model (1) has been recently investigated by the authors in [22, 23] and [20]. In [22] the existence of bounded variation solutions was analyzed by using variational methods and in [23] the existence of regular solutions was dealt with by means of classical phase plane and bifurcation techniques. The main result of [20] established the existence of a component of bounded variation solutions bifurcating from the trivial state $(\lambda, 0)$ in the special, but significant, case where $p = 2$. According to the results of these papers, it is already known that, under conditions (a_1) and (f_1), problem (1) cannot admit positive solutions if $\lambda < 0$ and that it possesses at least one positive bounded variation solution for sufficiently small $\lambda > 0$.

Quite strikingly, whether or not these bounded variation solutions are singular depends on whether or not condition (a_2) holds true: this is the main result of [21] which can be stated as follows.

Theorem 1. *Assume* (a_1) *and* (f_1). *Then, the following conclusions hold for sufficiently small* $\lambda > 0$*:*

(i) *any positive solution of* (1) *is singular if* (a_2) *holds;*

(ii) *any positive solution of* (1) *is regular if* (a_2) *fails.*

In other words, condition (a_2) completely characterizes, under (a_1) and (f_1), the development of singularities by the positive solutions of (1) for sufficiently small $\lambda > 0$.

By having a glance at condition (a_2) it becomes apparent that it fails whenever the function a is differentiable at the nodal point z, whereas a very simple example where (a_2) holds occurs when the function a is discontinuous at z, like, for instance, in the special case when a is assumed to be a positive constant, $A > 0$, in $[z - \eta_1, z)$ and a negative constant, $-B < 0$, in $(z, z + \eta_2]$, for some $\eta_1, \eta_2 > 0$. The huge contrast on the nature of the positive solutions of the problem with respect to the integrability properties of the function a near the node z can also be realized by considering any weight function a satisfying the requirements of (a_1) except for the fact that $a = 0$ in $[z - \eta, z + \eta]$ for some $\eta > 0$. In such case, thanks to the convexity and concavity properties of the positive bounded variation solutions of (1) guaranteed by [23, Prop. 3.6], any positive solution u must be linear in the interval $[z - \eta, z + \eta]$ and hence, due to (2), it cannot develop singularities.

As a consequence of Theorem 1, when $p = 2$, the global structure of the component of the positive solutions of (1), $\mathscr{C}_+$, whose existence is guaranteed by the main theorem of [20], drastically changes according to whether or not the condition (a_2) holds as illustrated in Figure 1, where λ_0 stands for the principal positive eigenvalue of the linear weighted problem

$$\begin{cases} -\varphi'' = \lambda a(x)\varphi & \text{in } (0,1), \\ \varphi'(0) = \varphi'(0) = 0. \end{cases}$$

The non-existence of positive regular solutions of (1) in the very special cases when $p = 2$ and the weight a is constant in $[0, z)$ and in $(z, 1]$ has been recently established in Section 8 of [23] by using some classical, but sophisticated, phase portrait techniques. This induced the authors to presume that an analogous non-existence result should also be valid for general weight functions a, without imposing the integrability condition (a_2). So, they formulated [23, Th. 7.1]. Theorem 1 in particular shows that [23, Th. 7.1] has to be complemented with condition (a_2).

Similarly as for $p = 2$, also in the case $p > 2$ the global structure of the set of positive solutions of (1), $\mathscr{C}_+$, whose existence is now guaranteed by [22, Th. 1.1] and [23, Th. 10.1], changes for sufficiently small $\lambda > 0$ according to whether or not condition (a_2) holds, as illustrated by Figure 2.

Our proof of Theorem 1 is based upon the characterization of the exact limiting profiles of the positive solutions of (1), both regular and singular, as the parameter λ approximates zero. These profiles are provided by the next theorem, regardless their particular nature.

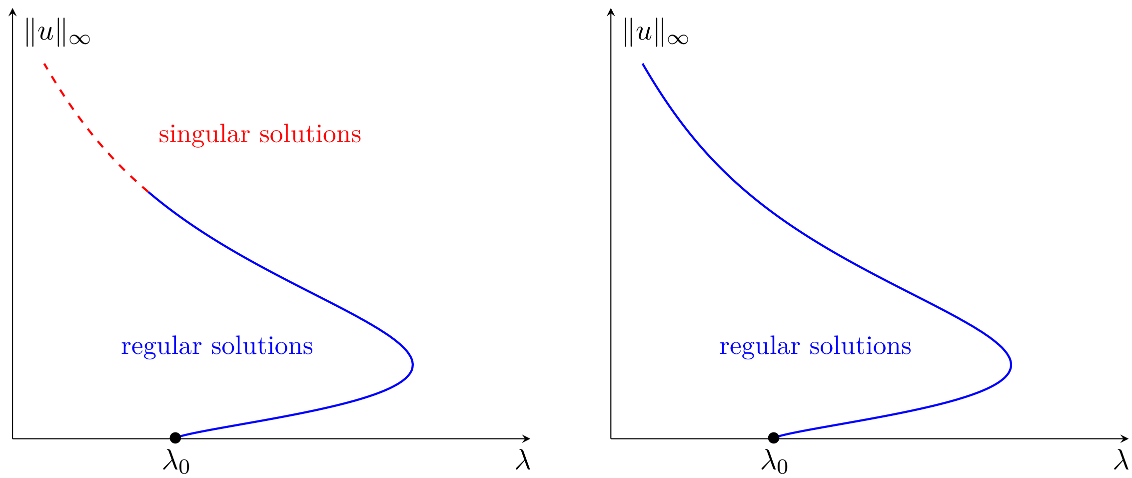

Figure 1. Global components emanating from the positive principal eigenvalue λ_0 in case $p = 2$ when (a_2) holds (on the left), or (a_2) fails (on the right).

Theorem 2. *Assume* (a_1) *and* (f_1), *and let* $((\lambda_n, u_n))_n$ *be an arbitrary sequence of positive solutions of* (1) *with* $\lim\limits_{n\to\infty} \lambda_n = 0$. *Then, for sufficiently small* $\eta > 0$, *the following assertions hold:*

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = 1 \ \ \textit{uniformly in} \ \ x \in [0, z-\eta],$$

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = \left(\frac{\int\limits_0^z a(t)\,dt}{-\int\limits_z^1 a(t)\,dt} \right)^{\frac{1}{q-1}} \ \ \textit{uniformly in} \ \ x \in [z+\eta, 1],$$

$$\lim_{n\to+\infty} (\lambda_n f(u_n(x))) = \frac{1}{\int\limits_0^z a(t)\,dt} \ \ \textit{uniformly in} \ \ x \in [0, z-\eta],$$

$$\lim_{n\to+\infty} (\lambda_n f(u_n(x))) = \frac{1}{-\int\limits_z^1 a(t)\,dt} \ \ \textit{uniformly in} \ \ x \in [z+\eta, 1],$$

$$\lim_{n\to+\infty} u_n'(x) = \frac{-\int\limits_0^x a(t)\,dt}{\sqrt{\Big(\int\limits_0^z a(t)\,dt\Big)^2 - \Big(\int\limits_0^x a(t)\,dt\Big)^2}} \ \ \textit{uniformly in} \ \ x \in [0, z-\eta],$$

and

$$\lim_{n\to+\infty} u_n'(x) = \frac{\int\limits_x^1 a(t)\,dt}{\sqrt{\Big(\int\limits_z^1 a(t)\,dt\Big)^2 - \Big(\int\limits_x^1 a(t)\,dt\Big)^2}} \ \ \textit{uniformly in} \ \ x \in [z+\eta, 1].$$

Note that condition (a_2) is equivalent to requiring the integrability in both intervals, $(0, z)$ and $(z, 1)$, of the asymptotic profile of the derivatives of the positive solutions of (1) as $\lambda \to 0^+$, which

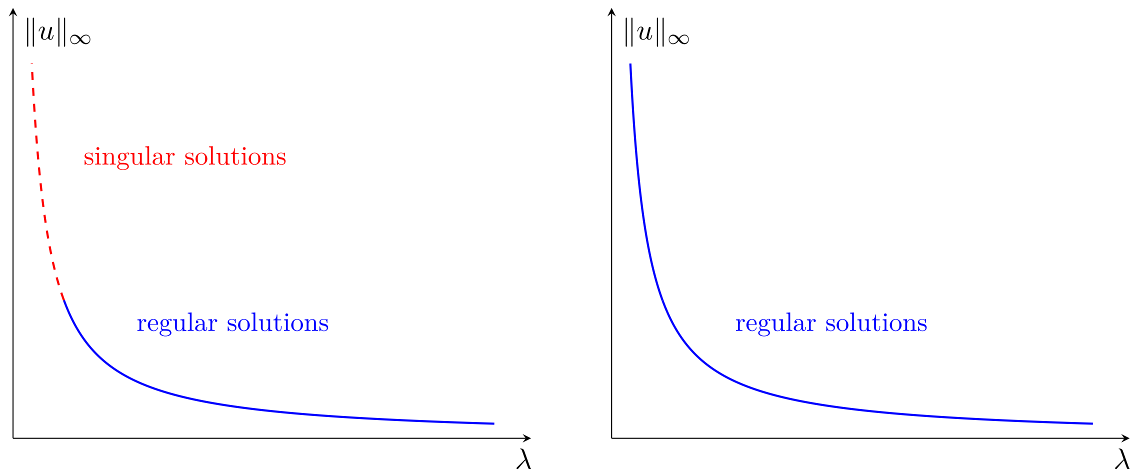

Figure 2. Global bifurcation diagrams in case $p > 2$ when (a_2) holds (on the left), or (a_2) fails (on the right).

is equivalent to impose that the "limiting derivative"

$$u'_\omega(x) = \begin{cases} \dfrac{-\int\limits_0^x a(t)\,dt}{\sqrt{\Big(\int\limits_0^z a(t)\,dt\Big)^2 - \Big(\int\limits_0^x a(t)\,dt\Big)^2}} & \text{for } \ x \in [0,z), \\ \dfrac{\int\limits_x^1 a(t)\,dt}{\sqrt{\Big(\int\limits_z^1 a(t)\,dt\Big)^2 - \Big(\int\limits_x^1 a(t)\,dt\Big)^2}} & \text{for } \ x \in (z,1], \end{cases}$$

belongs to both $L^1(0,z)$ and $L^1(z,1)$.

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[2] G. Anzellotti, The Euler equation for functionals with linear growth. *Trans. Amer. Math. Soc.* **290** (1985), no. 2, 483–501.

[3] E. Bombieri, E. De Giorgi and M. Miranda, Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. (Italian) *Arch. Rational Mech. Anal.* **32** (1969), 255–267.

[4] D. Bonheure, P. Habets, F. Obersnel and P. Omari, Classical and non-classical solutions of a prescribed curvature equation. *J. Differential Equations* **243** (2007), no. 2, 208–237.

[5] M. Burns and M. Grinfeld, Steady state solutions of a bi-stable quasi-linear equation with saturating flux. *European J. Appl. Math.* **22** (2011), no. 4, 317–331.

[6] P. Concus and R. Finn, On a class of capillary surfaces. *J. Analyse Math.* **23** (1970), 65–70.

[7] Ch. Corsato, C. De Coster and P. Omari, The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions. *J. Differential Equations* **260** (2016), no. 5, 4572–4618.

[8] Ch. Corsato, P. Omari and F. Zanolin, Subharmonic solutions of the prescribed curvature equation. *Commun. Contemp. Math.* **18** (2016), no. 3, 1550042, 33 pp.

[9] M. Emmer and Esistenza, unicità e regolarità nelle superfici de equilibrio nei capillari. (Italian) *Ann. Univ. Ferrara Sez. VII (N.S.)* **18** (1973), 79–94.

[10] R. Finn, The sessile liquid drop. I. Symmetric case. *Pacific J. Math.* **88** (1980), no. 2, 541–587.

[11] R. Finn, *Equilibrium Capillary Surfaces.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 284. Springer-Verlag, New York, 1986.

[12] C. Gerhardt, Boundary value problems for surfaces of prescribed mean curvature. *J. Math. Pures Appl. (9)* **58** (1979), no. 1, 75–109.

[13] C. Gerhardt, Global $C^{1,1}$-regularity for solutions of quasilinear variational inequalities. *Arch. Rational Mech. Anal.* **89** (1985), no. 1, 83–92.

[14] E. Giusti, Boundary value problems for non-parametric surfaces of prescribed mean curvature. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **3** (1976), no. 3, 501–548.

[15] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation.* Monographs in Mathematics, 80. Birkhäuser Verlag, Basel, 1984.

[16] E. Gonzalez, U. Massari and I. Tamanini, Existence and regularity for the problem of a pendent liquid drop. *Pacific J. Math.* **88** (1980), no. 2, 399–420.

[17] G. Huisken, Capillary surfaces over obstacles. *Pacific J. Math.* **117** (1985), no. 1, 121–141.

[18] A. Kurganov and Ph. Rosenau, On reaction processes with saturating diffusion. *Nonlinearity* **19** (2006), no. 1, 171–193.

[19] O. A. Ladyzhenskaya and N. N. Ural'tseva, Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations. *Comm. Pure Appl. Math.* **23** (1970), 677–703.

[20] J. López-Gómez and P. Omari, Global components of positive bounded variation solutions of a one-dimensional indefinite quasilinear Neumann problem. *Adv. Nonlinear Stud.* **19** (2019), no. 3, 437–473.

[21] J. López-Gómez and P. Omari, Characterizing the formation of singularities in a superlinear indefinite problem related to the mean curvature operator. *preprint*, 2019, 1–18.

[22] J. López-Gómez, P. Omari and S. Rivetti, Positive solutions of a one-dimensional indefinite capillarity-type problem: a variational approach. *J. Differential Equations* **262** (2017), no. 3, 2335–2392.

[23] J. López-Gómez, P. Omari and S. Rivetti, Bifurcation of positive solutions for a one-dimensional indefinite quasilinear Neumann problem. *Nonlinear Anal.* **155** (2017), 1–51.

[24] P. Rosenau, Free energy functionals at the high gradient limit. *Phys. Rev. A* **41** (1990), 2227–2230.

[25] J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables. *Philos. Trans. Roy. Soc. London Ser. A* **264** (1969), 413–496.

[26] R. Temam, Solutions généralisées de certaines équations du type hypersurfaces minima. (French) *Arch. Rational Mech. Anal.* **44** (1971/72), 121–156.