On the Behavior of Solutions with Positive Initial Data to Third Order Differential Equations with General Power-Law Nonlinearities

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1 Introduction

$$y''' = p(x, y, y', y'')|y|^{k_0}|y'|^{k_1}|y''|^{k_2}\operatorname{sgn}(yy'y''), \quad k_0, k_1, k_2 > 0, \tag{1.1}$$

with positive continuous and Lipschitz continuous in u, v, w function $p(x, u, v, w)$ satisfying inequalities

$$0 < m \le p(x, u, v, w) \le M < +\infty. \tag{1.2}$$

Equation (1.1) in the case $k_0 > 0$, $k_0 \ne 1$, $k_1 = k_2 = 0$, was studied by I. Astashova in [1, Chapters 6–8]. In particular, asymptotic classification of solutions to such equations was given in [4,6], and proved in [3].

For higher order differential equations, nonlinear with respect to derivatives of solutions, the asymptotic behavior of certain types of solutions was studied by V. M. Evtukhov, A. M. Klopot in [7,8]. Another approach to study asymptotic properties of solutions to higher order equations was offered by I. T. Kiguradze and T. A. Chanturia in [9].

Using methods described in [1,2,5] by I. V. Astashova, the behavior of solutions to (1.1) near domain boundaries is considered with respect to the values k_0, k_1 and k_2.

2 Main results

Consider positive increasing convex solutions to equation (1.1).

Theorem 2.1. *Suppose the function $p(x, u, v, w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*, and let $y(x)$ be a positive increasing convex on (x_1, x_2) solution to equation* (1.1)*. Then for $k_2 \ne 2$ the following estimates hold:*

$$m(y(x_1))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{2-k_2}}{2-k_2}\bigg|_{x_1}^{x_2} \le M(y(x_2))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2}, \tag{2.1}$$

and for $k_2 \ne 1$ the following estimates hold:

$$m(y'(x_1))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{1-k_2}}{1-k_2}\bigg|_{x_1}^{x_2} \le M(y'(x_2))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2}. \tag{2.2}$$

Proof. Let us prove inequalities (2.1). Since $y(x)$ is positive, increasing and convex, for $x \in [x_1, x_2]$ we have

$$m(y(x_1))^{k_0}(y'(x))^{k_1}(y''(x))^{k_2} \le y''' \le M(y(x_2))^{k_0}(y'(x))^{k_1}(y''(x))^{k_2},$$

hence

$$m(y(x_1))^{k_0}(y'(x))^{k_1}y'' \le y'''(y''(x))^{1-k_2} \le M(y(x_2))^{k_0}(y'(x))^{k_1}y''.$$

Let us integrate the above inequality on $[x_1,x_2]$:

$$m(y(x_1))^{k_0}\int\limits_{x_1}^{x_2}(y'(x))^{k_1}\,dy' \le \int\limits_{x_1}^{x_2}(y''(x))^{1-k_2}\,dy'' \le M(y(x_2))^{k_0}\int\limits_{x_1}^{x_2}(y'(x))^{k_1}\,dy',$$

so

$$m(y(x_1))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{2-k_2}}{2-k_2}\bigg|_{x_1}^{x_2} \le M(y(x_2))^{k_0}\,\frac{(y'(x))^{k_1+1}}{k_1+1}\bigg|_{x_1}^{x_2},$$

and thus, estimates (2.1) are obtained.

Now let us prove inequalities (2.2). Due to equation (1.1) and that fact that the function $p(x,u,v,w)$ is bounded, for any $x\in[x_1,x_2]$ it holds that

$$m(y'(x_1))^{k_1-1}(y(x))^{k_0}y'(x)(y''(x))^{k_2} \le y''' \le M(y'(x_2))^{k_1-1}(y(x))^{k_0}y'(x)(y''(x))^{k_2},$$

and therefore

$$m(y'(x_1))^{k_1-1}(y(x))^{k_0}y'(x) \le y'''(y''(x))^{-k_2} \le M(y'(x_2))^{k_1-1}(y(x))^{k_0}y'(x).$$

By integrating these inequalities on $[x_1,x_2]$, we obtain

$$m(y'(x_1))^{k_1-1}\int\limits_{x_1}^{x_2}(y(x))^{k_0}\,dy \le \int\limits_{x_1}^{x_2}(y''(x))^{-k_2}\,dy'' \le M(y'(x_2))^{k_1-1}\int\limits_{x_1}^{x_2}(y(x))^{k_0}\,dy,$$

which implies

$$m(y'(x_1))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2} \le \frac{(y''(x))^{1-k_2}}{1-k_2}\bigg|_{x_1}^{x_2} \le M(y'(x_2))^{k_1-1}\,\frac{(y(x))^{k_0+1}}{k_0+1}\bigg|_{x_1}^{x_2},$$

and estimates (2.2) are also proved. □

Theorem 2.2. *Suppose the function $p(x,u,v,w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*. Then the second derivative of any maximally extended solution $y(x)$ to equation* (1.1)*, satisfying the conditions $y(x_0)=y_0>0$, $y'(x_0)=y_1>0$, $y''(x_0)=y_2>0$ at some point x_0, tends to $+\infty$ as $x\to\widetilde{x}$, where $\widetilde{x}$ is the right domain boundary of solution $y(x)$, $x_0<\widetilde{x}\le+\infty$.*

Proof. Since initial data are positive and $p(x,u,v,w)>m$, we obtain $y'''(x)\ge my_0^{k_0}y_1^{k_1}y_2^{k_2}$ for $x\ge x_0$.

Denote $C_0=my_0^{k_0}y_1^{k_1}y_2^{k_2}$, then $y'''\ge C_0$, and by consequently integrating obtained inequalities on $[x_0,x]$ we derive

$$y''(x)>C_0(x-x_0),\quad y'(x)>\frac{C_0}{2}\,(x-x_0)^2,\quad y(x)>\frac{C_0}{6}\,(x-x_0)^3.$$

Then from equation (1.1) it follows that

$$y'''(x)>m\Big(\frac{C_0}{6}\,(x-x_0)^3\Big)^{k_0}\Big(\frac{C_0}{2}\,(x-x_0)^2\Big)^{k_1}(C_0(x-x_0))^{k_2}=\frac{mC_0^{k_0+k_1+k_2}}{6^{k_0}2^{k_1}}\,(x-x_0)^{3k_0+2k_1+k_2},$$

that is,

$$y''(x) > \widetilde{C}_0(x-x_0)^{3k_0+2k_1+k_2+1},$$

where $\widetilde{C}_0 > 0$ is a constant. Thus, $y''(x) \to +\infty$ as $x \to +\infty$, and the theorem is proved for $\widetilde{x} = +\infty$.

Consider now the case $\widetilde{x} < +\infty$. If for a constant $D > 0$ inequality $y''(x) \le D$ holds for $x \in (x_0, \widetilde{x})$, then

$$\begin{aligned}
y'(x) &\le D(x-x_0) + y'(x_0) \le D(\widetilde{x}-x_0) + y'(x_0) = D_1 < +\infty,\\
y(x) &\le D_1(x-x_0) + y(x_0) \le D_1(\widetilde{x}-x_0) + y'(x_0) = D_2 < +\infty,
\end{aligned}$$

so $y'''(x) \le MD_2^{k_0}D_1^{k_1}D^{k_2} < +\infty$, and, since the solution and all its derivatives up to the third are increasing and bounded on a finite interval, there exist finite limits of the solution and its derivatives as $x \to \widetilde{x}$. Then the solution $y(x)$ can be extended to the right of $\widetilde{x}$, and we obtain a contradiction.

Thus, $y''(x) \to +\infty$ as $x \to \widetilde{x}$, and the theorem is proved. $\square$

Theorem 2.3. *Suppose $k_0 + k_1 + k_2 > 1$, and the function $p(x,u,v,w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2). *Then for any maximally extended solution $y(x)$ to equation* (1.1), *satisfying the conditions $y(x_0) \ge 0$, $y'(x_0) \ge 0$, $y''(x_0) = y_2 > 0$ at some point x_0, its right domain boundary $\widetilde{x}$ is finite and satisfies the estimate*

$$\widetilde{x} - x_0 < \xi y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}},$$

with $\xi = \big(\frac{(2k_0+k_1+1)2^{k_0}}{m}\big)^{\frac{1}{2k_0+k_1+1}}\big(1 - 2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}\big)^{-1}$.

Proof. As it was shown above, the second derivative of such solution is infinitely increasing as argument tends to the right domain boundary. Consider the sequence of points x_i, $i = 0,1,\dots$, such that $y''(x_i) = 2y''(x_{i-1}) = 2^i y_2$.

For $x \in [x_i, x_{i+1}]$ the following inequalities hold:

$$\begin{aligned}
y''(x) &\ge 2^i y_2,\\
y'(x) > y'(x) - y'(x_i) &\ge 2^i y_2(x-x_i),\\
y(x) > y(x) - y(x_i) &\ge 2^{i-1}\, y_2(x-x_i)^2.
\end{aligned}$$

Then from equation (1.1) we derive

$$\begin{aligned}
y'''(x) &> m\big|2^{i-1}y_2(x-x_i)^2\big|^{k_0}\big|2^i y_2(x-x_i)\big|^{k_1}|2^i y_2|^{k_2},\\
y'''(x) &> m\cdot 2^{i(k_0+k_1+k_2)-k_0} y_2^{k_0+k_1+k_2}(x-x_i)^{2k_0+k_1}.
\end{aligned}$$

By integrating this inequality on $[x_i, x_{i+1}]$, we obtain

$$\begin{aligned}
2^{i+1}y_2 - 2^i y_2 &> \frac{m\cdot 2^{i(k_0+k_1+k_2)-k_0}}{2k_0+k_1+1}\, y_2^{k_0+k_1+k_2}(x_{i+1}-x_i)^{2k_0+k_1+1},\\
2^i y_2^{-(k_0+k_1+k_2-1)} &> \frac{m\cdot 2^{i(k_0+k_1+k_2)-k_0}}{2k_0+k_1+1}\,(x_{i+1}-x_i)^{2k_0+k_1+1},\\
(x_{i+1}-x_i)^{2k_0+k_1+1} &< \frac{(2k_0+k_1+1)\cdot 2^{k_0}}{m}\,(2^i y_2)^{-(k_0+k_1+k_2-1)},
\end{aligned}$$

and, since $2k_0 + k_1 + 1 > 0$,

$$x_{i+1} - x_i < \Big(\frac{(2k_0 + k_1 + 1)2^{k_0}}{m}\Big)^{\frac{1}{2k_0+k_1+1}} (2^i y_2)^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}.$$

Now let us summarize these inequalities:

$$\sum_{i=0}^{+\infty}(x_{i+1} - x_i) < \Big(\frac{(2k_0 + k_1 + 1)2^{k_0}}{m}\Big)^{\frac{1}{2k_0+k_1+1}} y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}} \sum_{i=0}^{+\infty} 2^{-i\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}.$$

Since $k_0 + k_1 + k_2 > 1$, the series in the right part converges and

$$\widetilde{x} - x_0 = \lim_{i\to+\infty} x_i - x_0 = \sum_{i=0}^{+\infty}(x_{i+1} - x_i) < \xi y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}},$$

with $\xi = \big(\frac{(2k_0+k_1+1)2^{k_0}}{m}\big)^{\frac{1}{2k_0+k_1+1}} (1 - 2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}})^{-1}$.

Thus, $\widetilde{x}$ is finite and the theorem is proved. □

Theorem 2.4. *Suppose $k_0+k_1+k_2 \neq 1$, $k_2 \neq 1$, $k_2 \neq 2$, and the function $p(x, u, v, w)$ is continuous, Lipschitz continuous in u, v, w, and satisfies inequalities* (1.2)*. Let $y(x)$ be a maximally extended solution to equation* (1.1)*, satisfying the conditions $y(x_0) \geq 0$, $y'(x_0) \geq 0$, $y''(x_0) > 0$ at some point x_0. Then*

1. *if $k_0 + k_1 + k_2 < 1$, then $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < +\infty$ or $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} = +\infty$;*
2. *if $k_0 + k_1 + k_2 > 1$, $k_1 \leq 1$, $k_2 < 1$, then $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$;*
3. *if $k_1 > 1$, $k_2 < 1$, then $y \to const$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$ or $y \to +\infty$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$;*
4. *if $1 < k_2 < 2$, then $y \to const$, $y' \to +\infty$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$;*
5. *if $k_2 > 2$, then $y \to const$, $y' \to const$, $y'' \to +\infty$ as $x \to \widetilde{x} < \infty$.*

Proof. Since the initial data are nonnegative as well as the function $p(x, u, v, w)$, solution $y(x)$ and its first, second and third derivatives are positive and increasing as $x \to \widetilde{x}$, where $\widetilde{x}$ is a right domain boundary of $y(x)$. According to the Theorem 2.2, the second derivative is increasing and unbounded.

Let us show that if the first derivative is bounded, then the solution with positive initial data cannot be bounded. Indeed, let $y' \leq C$, then $y \leq C(x - x_0) + y(x_0)$, which implies that in the case $\widetilde{x} < +\infty$ the solution is also bounded. If the solution is infinitely extensible to the right, then, since $y'(x_0) > 0$, we derive $y(x) > y'(x_0)(x - x_0) + y(x_0)$, and unboundedness of this solution follows from unboundedness of x.

Thus, there are three possible options: a solution and its first derivative are bounded; a solution is bounded, but its derivative is unbounded, and both solution and its derivative are unbounded.

At first, let us consider the case $k_2 > 2$. In this case $k_0 + k_1 + k_2 > 1$, and by Theorem 2.3, the domain of solution is finite. Values $k_0 + 1$ and $k_1 + 1$ are positive; besides, $1 - k_2 < 2 - k_2 < 0$, and therefore, using inequality (2.1) on the interval (x_0, x), as $x \to \widetilde{x}$ we have

$$m(y(x_0))^{k_0} \frac{(y'(x))^{k_1+1} - (y'(x_0))^{k_1+1}}{k_1 + 1} \leq \frac{(y''(x))^{2-k_2} - (y''(x_0))^{2-k_2}}{2 - k_2} < +\infty,$$

which implies that $y'(x)$ is bounded as $x \to \widetilde{x}$. Analogously, inequality (2.1) implies that the solution $y(x)$ is also bounded.

Consider the case $1 < k_1 < 2$. Again, $k_0 + k_1 + k_2 > 1$, and, by Theorem 2.3, the domain of $y(x)$ is finite; also $1 - k_2 < 0 < 2 - k_2$, and, due to (2.1), (2.2) and the fact that $y'' \to +\infty$ as $x \to \widetilde{x}$, we derive

$$M(y(x))^{k_0} \, \frac{(y'(x))^{k_1+1} - (y'(x_0))^{k_1+1}}{k_1+1} \geq \frac{(y''(x))^{2-k_2} - (y''(x_0))^{2-k_2}}{2-k_2} \to +\infty,$$
$$m(y'(x_0))^{k_1-1} \, \frac{(y(x))^{k_0+1} - (y(x_0))^{k_0+1}}{k_0+1} \leq \frac{(y''(x))^{1-k_2} - (y''(x_0))^{1-k_2}}{1-k_2} < +\infty,$$

hence $y(x) \to const$ and $y'(x) \to +\infty$ as $x \to \widetilde{x}$.

Further, suppose $k_2 < 1$, $k_1 > 1$. Then $k_0 + k_1 + k_2 > 1$, the domain of solution is finite, $2 - k_2 > 1 - k_2 > 0$, and we obtain

$$M(y(x))^{k_0} \, \frac{(y'(x))^{k_1+1} - (y'(x_0))^{k_1+1}}{k_1+1} \geq \frac{(y''(x))^{2-k_2} - (y''(x_0))^{2-k_2}}{2-k_2} \longrightarrow +\infty,$$
$$M(y'(x))^{k_1-1} \, \frac{(y(x))^{k_0+1} - (y(x_0))^{k_0+1}}{k_0+1} \geq \frac{(y''(x))^{1-k_2} - (y''(x_0))^{1-k_2}}{1-k_2} \longrightarrow +\infty.$$

In this case there are two possible options: $y \to const$, $y' \to +\infty$, and $y \to +\infty$, $y' \to \ +\infty$.

Finally, for $k_2 < 1$, $k_1 \leq 1$, according to the above inequalities, the only possible option is $y \to +\infty$, $y' \to \ +\infty$; moreover, if $k_0 + k_1 + k_2 > 1$, then $\widetilde{x} < +\infty$, and the theorem is proved. $\square$

Remark 2.1. In the cases 1 and 3 Theorem 2.4 does not state the existence of solutions of every possible type of behavior. In the cases 4 and 5 for $k_0 \geq 1$, $k_1 \geq 1$, $k_2 > 1$ the existence of solutions of described type is guaranteed by classical existence and uniqueness theorem. For $0 < k_0 < 1$, $k_1 \geq 1$, $k_2 \geq 1$ the existence of solutions to equation (1.1) with positive initial data is guaranteed by the following theorem.

Theorem 2.5 (I. Astashova, [1]). *Let the function $p(x, y_0, \dots, y_{n-1})$ be continuous in x and Lipschitz continuous in $y_0, \dots, y_{n-1}$. Then for any set of numbers $x_0, y_0^0, \dots, y_{n-1}^0$ with not every y_i^0 equal to zero, the corresponding Cauchy problem for the equation*

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sgn} y, \quad n \geq 2, \quad 0 < k < 1,$$

has a unique solution.

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