Solvability of the Boundary Value Problem for One Class of Higher-Order Nonlinear Partial Differential Equations

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In the Euclidean space $\mathbb{R}^{n+1}$ of the variables $x=(x_1,x_2,\dots,x_n)$ and t we consider the nonlinear equation of the type

$$L_f u := \frac{\partial^{2(2k+1)}u}{\partial t^{2(2k+1)}} - \Delta^2 u + f(u,\nabla u) = F(x,t), \tag{1}$$

where f and F are given, and u is an unknown real functions, $\nabla := (\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n},\frac{\partial}{\partial t})$, $\Delta := \sum\limits_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$, k is a natural number and $n\geq 2$.

For the equation (1) we consider the boundary value problem: find in the cylindrical domain $D_T := \Omega\times(0,T)$, where Ω is a Lipschitz domain in $\mathbb{R}^n$, a solution $u=u(x,t)$ of that equation according to the boundary conditions

$$\frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T} = 0, \quad i=0,\dots,2k, \tag{2}$$

$$u\big|_{\Gamma_T} = 0, \quad \frac{\partial u}{\partial\nu}\Big|_{\Gamma_T} = 0, \tag{3}$$

where $\Gamma_T := \partial\Omega\times(0,T)$ is the lateral face of the cylinder D_T, $\Omega_0:\ x\in\Omega,\ t=0$ and $\Omega_T:\ x\in\Omega,\ t=T$ are bottom and top bases of this cylinder, respectively, and $\frac{\partial}{\partial\nu}$ is a derivative along the outer normal to the boundary ∂D_T of the domain D_T. For $T=\infty$ we have $D_\infty=\Omega\times(0,\infty)$, $\Gamma_\infty=\partial\Omega\times(0,\infty)$.

Note that the linear part of the operator L_f from (1), i.e. L_0 is a hypoelliptic operator.

Below, for function $f=f(s_0,s_1,\dots,s_{n+1}),(s_0,s_1,\dots,s_{n+1})\in\mathbb{R}^{n+2}$ we assume that

$$f\in C(\mathbb{R}^{n+2}) \tag{4}$$

and

$$\big|f(s_0,s_1,\dots,s_{n+1})\big| \leq M + \sum_{i=0}^{n+1} M_i|s_i|^{\alpha_i} \ \ \forall\, s=(s_0,s_1,\dots,s_{n+1})\in\mathbb{R}^{n+2}, \tag{5}$$

where $M, M_i, \alpha_i = const > 0$, $i=0,1,\dots,n+1$.

Denote by $C^{4,4k+2}(\overline{D}_T)$ the space of continuous functions in $\overline{D}_T$ having continuous partial derivatives $\partial_x^\beta u$, $\frac{\partial^l u}{\partial t^l}$ in $\overline{D}_T$, where $\partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}$, $\beta=(\beta_1,\dots,\beta_n)$, $|\beta|=\sum\limits_{i=1}^{n}\beta_i\leq 4$; $l=1,\dots,4k+2$.

Assume

$$C_0^{4,4k+2}(\overline{D}_T,\partial D_T) := \bigg\{u\in C^{4,4k+2}(\overline{D}_T):\ u\big|_{\Gamma_T} = \frac{\partial u}{\partial\nu}\Big|_{\Gamma_T} = 0,\ \frac{\partial^i u}{\partial t^i}\Big|_{\Omega_0\cup\Omega_T} = 0,\ i=0,\dots,2k\bigg\}.$$

Let $u \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$ be a classical solution of the problem $(1), (2), (3)$. Multiplying both parts of the equation (1) by an arbitrary function $\varphi \in C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$ and integrating the obtained equation by parts over the domain D_T, we obtain

$$-\int\limits_{D_T} \Big[\frac{\partial^{2k+1}u}{\partial t^{2k+1}} \cdot \frac{\partial^{2k+1}\varphi}{\partial t^{2k+1}} + \Delta u \cdot \Delta\varphi\Big]\, dx\, dt + \int\limits_{D_T} f(u, \nabla u)\varphi\, dx\, dt \\ = \int\limits_{D_T} F\varphi\, dx\, dt \ \ \forall\, \varphi \in C^{4,4k+2}(\overline{D}_T, \partial D_T). \tag{6}$$

We take the equality (6) as a basis for our definition of the weak generalized solution u of the problem $(1), (2), (3)$.

Introduce the Hilbert space $W_0^{2,2k+1}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{W_0^{2,2k+1}(D_T)} = \int\limits_{D_T} \bigg[u^2 + \sum_{i=1}^{n} \Big(\frac{\partial u}{\partial x_i}\Big)^2 + \sum_{i,j=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \sum_{i=1}^{2k+1} \Big(\frac{\partial^i u}{\partial t^i}\Big)^2\bigg]\, dx\, dt \tag{7}$$

of the classical space $C_0^{4,4k+2}(\overline{D}_T, \partial D_T)$.

Remark 1. From (7) it follows that if $u \in W_0^{2,2k+1}(D_T)$, then $u \in \overset{\circ}{W}{}_2^1(D_T)$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}, \frac{\partial^l u}{\partial t^l} \in L_2(D_T)$; $i, j = 1, \dots, n$; $l = 1, \dots, 2k+1$. Here $W_2^m(D_T)$ is the well-known Sobolev space consisting of the elements of $L_2(D_T)$, having generalized derivatives from $L_2(D_T)$ up to m-th order inclusively, and $\overset{\circ}{W}{}_2^1(D_T) = \{u \in W_2^1(D_T) :\ u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T} = 0$ is understood in the sense of the trace theory. Moreover, when the domain Ω is convex, and therefore the domain D_T is also convex, and since the following estimate

$$\int\limits_{D_T} \bigg[\sum_{i,j=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial x_j}\Big)^2 + \sum_{i=1}^{n} \Big(\frac{\partial^2 u}{\partial x_i \partial t}\Big)^2 + \Big(\frac{\partial^2 u}{\partial t^2}\Big)^2\bigg]\, dx\, dt \\ \le c \int\limits_{D_T} \Big[\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{\partial^2 u}{\partial t^2}\Big]^2\, dx\, dt \ \ \forall\, u \in \overset{\circ}{C}2(\overline{D}_T, \partial D_T) := \Big\{u \in C^2(\overline{D}_T) :\ u\big|_{\partial D_T} = 0\Big\}$$

holds with a positive constant c not dependant on u and the domain D_T, then from (7) we have continuous embedding of spaces

$$W_0^{2,2k+1}(D_T) \subset W_2^2(D_T). \tag{8}$$

Below, we assume that Ω is a convex domain.

Remark 2. As it is known the space $W_2^2(D_T)$ is continuously and compactly embedded into $L_p(D_T)$ for $p < \frac{2(n+1)}{n-3}$ when $n > 3$ and for any $p \ge 1$ when $n = 2, 3$; analogously, the space $W_2^1(D_T)$ is continuously and compactly embedded into $L_q(D_T)$ if $q < \frac{2(n+1)}{n-1}$. Therefore, taking into account continuous embedding of the spaces (8), the inequality (5) and the properties of the Nemytski operators N_i, $i = 0, 1, \dots, n+1$, acting by formula $N_i v = |v|^{\alpha_i}$, we get that the nonlinear operator $N : W_0^{2,2k+1}(D_T) \to L_2(D_T)$ acting by formula $Nu = f(u, \Delta u)$, will be continuous and compact if the nonlinearity exponent α_i in the right-hand side of the inequality (5) satisfies the following inequalities:

$$1 < \alpha_0 < \frac{n+1}{n-3} \ \text{ for } \ n > 3; \quad \alpha_0 > 1 \ \text{ for } \ n = 2,3; \tag{9}$$

$$1 < \alpha_i < \frac{n+1}{n-1}\,, \ \ i = 1,\dots,n+1, \ \ n \geq 2. \tag{10}$$

Besides, from the above-mentioned remarks it follows that if $u \in \overset{\circ}{W}{}^{2,2k+1}_0(D_T)$, then $f(u,\nabla u) \in L_2(D_T)$ and for $u_m \to u$ in the space $W^{2,2k+1}_0(D_T)$ we have $f(u_m,\nabla u_m) \to f(u,\nabla u)$ in the space $L_2(D_T)$.

Definition 1. Let function f satisfy the conditions (4), (5), (9) and (10); $F \in L_2(D_T)$. The function $u \in W^{2,2k+1}_0(D_T)$ is said to be a weak generalized solution of the problem (1), (2), (3) if for any $\varphi \in W^{2,2k+1}_0(D_T)$ the integral equality (6) is valid.

Notice that when the conditions (4), (5), (9) and (10) are fulfilled, if $u \in W^{2,2k+1}_0(D_T)$ and $\varphi \in W^{2,2k+1}_0(D_T)$, then according to Remark 2 we have $f(u,\nabla u) \in L_2(D_T)$, $\varphi \in L_2(D_T)$ and the second addend

$$\int\limits_{D_T} f(u,\nabla u)\varphi \, dx \, dt$$

in the left-hand side of the equality (6) is defined correctly.

It is not difficult to verify that if the solution of the problem (1), (2), (3) in the sense of Definition 1 belongs to the class $C^{4,4k+2}_0(\overline{D}_T,\partial D_T)$, then it will also be a classical solution of this problem.

Definition 2. Let function f satisfy the conditions (4), (5), (9) and (10); $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ $\forall\, T > 0$. We say that the problem (1), (2), (3) is globally solvable in the class $W^{2,2k+1}_0$ if for any $T > 0$ this problem has at least one weak generalized solution $u \in W^{2,2k+1}_0(D_T)$ in the sense of Definition 1.

Definition 3. Let function f satisfy the conditions (4), (5), (9) and (10); $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ $\forall\, T > 0$. We say that the problem (1), (2), (3) is locally solvable in the class $W^{2,2k+1}_0$ if there exists a number $T_0 = T_0(F)$ such that for any positive $T < T_0$ this problem has at least one weak generalized solution $u \in W^{2,2k+1}_0(D_T)$ in the sense of Definition 1.

It is proved that when the conditions (4), (5), (9) and (10); $F \in L_{2,loc}(D_\infty)$ and $F|_{D_T} \in L_2(D_T)$ $\forall\, T > 0$ are fulfilled, then the problem (1), (2), (3) is locally solvable in the class $W^{2,2k+1}_0$ in the sense of Definition 3, and for some additional conditions on the problem's data, in certain cases the problem (1), (2), (3) is locally solvable whereas it is not globally solvable, and in other cases we have a global solvability in the sense of Definition 2.

The case of uniqueness of the solution of this problem in D_∞ is also considered.