Regularization Method in Stability Analysis of Stochastic Functional Differential Equations

Ramazan I. Kadiev

Dagestan Research Center of the Russian Academy of Sciences & Department of Mathematics, Dagestan State University, Makhachkala, Russia

E-mail: kadiev_r@mail.ru

Arcady Ponosov

Department of Mathematical Sciences and Technology, Norwegian University of Life Sciences, P. O. Box 5003, N-1432 Ås, Norway

E-mail: arkadi@nmbu.no

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By E we denote the expectation on this probability space. By $Z := (z_1, \dots, z_m)^T$ we denote an m-dimensional semimartingale (see, e.g. [7]). A popular example of such Z is the vector Brownian motion (the Wiener process). The linear space k^n consists of all n-dimensional $\mathcal{F}_0$-measurable random variables.

The main idea of the method, which is outlined below, is to represent the property of Lyapunov stability in terms of invertibility of certain linear operators in suitable functional spaces.

The following linear homogeneous stochastic delay differential equation is considered

$$dx(t) = (V_h x)(t)\, dZ(t) \ \ (t \geq 0) \tag{1}$$

endowed with two initial conditions

$$x(s) = \varphi(s) \ \ (s < 0) \tag{2}$$

and

$$x(0) = b. \tag{3}$$

Here V_h is a k-linear Volterra operator which is defined in certain linear spaces of vector stochastic processes, φ is an $\mathcal{B}(-\infty, 0) \otimes \mathcal{F}_0$-measurable stochastic process and $b \in k^n$. By k-linearity of the operator V_h we mean the following property

$$V_h(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 V_h x_1 + \alpha_2 V_h x_2,$$

which holds for all $\mathcal{F}_0$-measurable, bounded and scalar random values α_1, α_2 and all stochastic processes x_1, x_2 belonging to the domain of the operator V_h. The exact assumptions on the domain and the range of V_h are specified below in connection with the properties of the associated operator V.

The solution of the initial value problem (1)–(3) will be denoted by $x(t, b, \varphi)$, $t \in (-\infty, \infty)$. The solution is always assumed to exist and to be unique for an appropriate choice of $\varphi(s), b$: for specific conditions see e.g. [3].

According to the habilitation thesis [3], the following classes of linear stochastic equations are particular cases of Eq. (1):

(A) *Systems of linear ordinary (i.e. non-delay) stochastic differential equations driven by an arbitrary semimartingale (in particular, systems of ordinary Itô equations)*;

(B) *Systems of linear stochastic differential equations with discrete delays driven by a semimartingale (in particular, systems of Itô equations with discrete delays)*;

(C) *Systems of linear stochastic differential equations with distributed delays driven by a semimartingale (in particular, systems of Itô equations with distributed delays)*;

(D) *Systems of linear stochastic integro-differential equations driven by a semimartingale (in particular, systems of Itô integro-differential equations)*;

(E) *Systems of linear stochastic functional difference equations driven by a semimartingale (in particular, systems of Itô functional difference equations).*

Definition 1. For a given real number q $(1 \le q < \infty)$ we call the zero solution of Eq. (1)

- q-stable (with respect to the initial data b and φ) if for any $\varepsilon > 0$ there is $\delta(\varepsilon) > 0$ such that $E|b|^q + \operatorname*{ess\,sup}\limits_{s<0} E|\varphi(s)|^q < \delta$ implies $E|x(t,b,\varphi)|^q \le \varepsilon$ for all $t \ge 0$ and all $\mathcal{F}_0$-measurable φ, b;
- exponentially q-stable if there exist positive constants K, λ such that the inequality
$$E|x(t,b,\varphi)|^q \le K\big(E|b|^q + \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^q\big)\exp\{-\lambda s\}$$
holds true for all $t \ge 0$ and all $\mathcal{F}_0$-measurable φ, b.

Let S^n be a linear subspace of the space of $\mathcal{F}_t$-adapted, n-dimensional stochastic processes whose trajectories belong to a normed space E with the norm $\|\cdot\|_E$. Then we denote by S^n_q $(1 \le q < \infty)$, the linear subspace of S^n containing all processes $f \in S^n$, for which the norm defined by $\|f\|^q_{S^n_q} = E\|f\|^q_E$ is finite.

For instance, if Φ^n stands for all $\mathcal{F}_0$-measurable, n-dimensional prehistory functions φ with essentially bounded trajectories, then the norm in Φ^n_q is given by
$$\|\varphi\|_q = \operatorname*{ess\,sup}_{s<0} E|\varphi(s)|^q.$$
This simplifies the notation in Definition 1, where the expression $E|b|^q + \operatorname{ess\,sup}_{s<0} E|\varphi(s)|^q$ may be replaced by $\|b\|^q_{k^n_q} + \|\varphi\|^q_{\Phi^n_q}$.

To describe the regularization method, one needs to represent (1)–(2) in a canonical form [1,3]. Let $x(t)$ be a stochastic process on $[0,+\infty)$ and $x_+(t)$ be a stochastic process on $(-\infty,+\infty)$ coinciding with $x(t)$ for $t \ge 0$ and equalling 0 for $t < 0$, while $\varphi_-(t)$ be a stochastic process on $(-\infty,+\infty)$ coinciding with $\varphi(t)$ for $t < 0$ and equalling 0 for $t \ge 0$. Then the stochastic process $x_+(t) + \varphi_-(t)$, defined for $t \in (-\infty,+\infty)$ will be a solution of the problem (1)–(3) if $x(t)$ $(t \in [0,+\infty))$ satisfies the initial value problem
$$dx(t) = \big[(Vx)(t) + f(t)\big]\,dZ(t) \ \ (t \ge 0), \tag{4}$$
$$x(0) = b, \tag{5}$$
where $(Vx)(t) := (V_h x_+)(t)$, $f(t) := (V_h \varphi_-)(t)$ for $t \ge 0$. Indeed, by k-linearity we have that $V_h(x_+ + \varphi_-) = V_h(x_+) + V_h(\varphi_-) = Vx + f$, which gives (4). Note that f is uniquely defined by the prehistory function φ. Let us also observe that the initial value problem (4)–(5) is equivalent to the initial value problem (1)–(3) only for f, which have the representation $f = V_h\varphi'$, where φ' is an arbitrary extension of the function φ to the real line $(-\infty,\infty)$.

The solution of (4)–(5) is below denoted by $x_f(t,b)$.

Let B^n be a linear subspace of the space of $\mathcal{F}_t$-adapted stochastic processes with trajectories which are almost surely essentially bounded on $[0,\infty)$. According to our notation, the norm in the space B^n_q is defined by

$$\|f\|^q_{B^n_q} = \operatorname*{ess\,sup}_{t\geq 0} E|f(t)|^q.$$

Let $L^n(Z)$ be the set of all $n\times m$-matrix predictable stochastic processes defined on $[0,+\infty)$ and whose rows are locally integrable with respect to the semimartingale Z, see e.g. [3], and D^n be the set of all n-dimensional stochastic processes on $[0,+\infty)$, which can be represented as

$$x(t) = x(0) + \int\limits_0^t H(s)\,dZ(s),$$

where $x(0)\in k^n$, $H\in L^n(Z)$. The space D^n and its linear subspaces D^n_q are called the spaces of solutions of Eq. (4) (see [3]). The operator V is usually assumed to be a bounded linear operator from D^n_q to $L^n_q(Z)$ for some $1\leq q<\infty$.

This yields two linear operators

$$\mathcal{L}_1 : \varphi \longmapsto (V_h\varphi_-)(t) \tag{6}$$

and

$$\mathcal{L}_2 : f \longmapsto x_f(\,\cdot\,,b). \tag{7}$$

The following result is crucial for the framework (see e.g. [5]).

Theorem 1. *Assume that the linear operators $\mathcal{L}_1 : \Phi_q \to B^n_q$ and $\mathcal{L}_2 : B^n_q \to D^n_q$ are bounded. Then the zero solution of Eq.* (1) *is q-stable in the sense of Definition 1.*

In applications, the operator $\mathcal{L}_1$ is usually bounded, so that the only challenge in application of Theorem 1 is to prove boundedness of the operator $\mathcal{L}_2$. This can be done by the regularization method called in [1] and [3] 'the W-method'. The regularization is usually constructed with the help of an auxiliary equation

$$dx(t) = \big[(Qx)(t) + g(t)\big]\,dZ(t)\ \ (t\geq 0), \tag{8}$$

where Q is again a k-linear Volterra operator. This equation is similar to Eq. (4), possesses the existence and uniqueness property as well, but it is usually chosen to be 'simpler' in the sense that the required stability property for this equation is already known (see assumption (2) in Theorem 2 below).

The following representation formula for the solutions of Eq. (8) can be directly deduced from the existence and uniqueness property

$$x(t) = U(t)x(0) + (Wg)(t)\ \ (t\geq 0), \tag{9}$$

where $U(t)$ is the fundamental matrix of the associated homogeneous equation, and W is the corresponding Cauchy operator.

Using representation (9) we can regularize Eq. (4). This algorithm is based on the framework described in [3,5].

Using Eq. (8) we rewrite Eq. (4) as follows

$$dx(t) = \big[(Qx)(t) + ((V-Q)x)(t) + f(t)\big]\,dZ(t)\ \ (t\geq 0),$$

or, taking (9) into account, as

$$x(t)=U(t)x(0)+(W(V-Q)x)(t)+(Wf)(t)\ \ (t\geq 0).$$

Putting $W(V-Q)=\Theta$, we obtain the operator equation

$$x(t)=(\Theta x)(t)+U(t)x(0)+(Wf)(t)\ \ (t\geq 0). \tag{10}$$

Theorem 2. *Assume that Eq.* (4) *and the reference equation* (8) *satisfy the following conditions:*

(1) *the linear operators V, Q act continuously from D_q^n to B_q^n;*

(2) *the Cauchy operator W in* (9) *constructed for the reference equation* (8) *is bounded as an operator from B_q^n to D_q^n;*

(3) *the operator $I-\Theta: D_q^n\to D_q^n$ has a bounded inverse.*

Then the operator $\mathcal{L}_2: B_q^n\to D_q^n$ in (7) *is bounded.*

Theorems 1 and 2 justify the regularization method for Lyapunov stability of stochastic linear functional differential equations. The main challenge of the method is to prove that the operator $I-\Theta$ has a bounded inverse. In [3–5] (see also the references therein) this property is checked by estimating the norm of the integral operator Θ: if $\|\Theta\|_{D_q^n}<1$ in the inequality

$$\|x\|_{D_q^n}\leq\|\Theta\|_{D_q^n}\|x\|_{D_q^n}+K_1\|x(0)\|_{k_q^n}+K_2\|f\|_{B_q^n}, \tag{11}$$

then Eq. (1) is q-stable due to Theorem 1. Moreover, if $q\geq 2$ and the equation remains q-stable after the substitution $y(t)=\exp(\lambda t)x(t)$ for some positive λ, then Eq. (1) is, in fact, exponentially q-stable.

Another approach, which has recently been suggested in [2] in the deterministic case and in [6] in the stochastic case, is based on the properties of monotone operators. In this case, the estimation is done componentwise, and if the resulting matrix has a bounded inverse, then one still obtains inequalities like (11). A short description of this method is given below.

Recall that an $m\times m$-matrix $B=(b_{ij})_{i,j=1}^m$ is said to be nonnegative, resp. positive if $b_{ij}\geq 0$, resp. $b_{ij}>0$ for all $i,j=1,\dots,m$.

Definition 2. A matrix $\Gamma=(\gamma_{ij})_{i,j=1}^n$ is called a (non-singular) $\mathcal{M}$-matrix if $\gamma_{ij}\leq 0$ for $i,j=1,\dots,n$, $i\neq j$, and all the principal minors of the matrix Γ are positive.

Let

$$x(t)=\operatorname{col}(x_1(t),\dots,x_n(t)),\ \ \overline{x}_i=\sup_{t\geq 0}\big(E|x_i(t)|^q\big)^{1/q},\ \ \overline{x}=\operatorname{col}(\overline{x}_1,\dots,\overline{x}_n).$$

Suppose that after componentwise estimation in the vector equation (10) we get the following vector inequality

$$D\overline{x}\leq\|x(0)\|_{k_q^n}\overline{e}_1+\|f\|_{B_q^n}\overline{e}_2, \tag{12}$$

where D is an $n\times n$-matrix, $\overline{e}_1$, $\overline{e}_2$ are some column n-vectors with nonnegative components. Typically, $D=\overline{E}-T$, where $\overline{E}$ is the $n\times n$ identity matrix, while T and $\overline{e}_i$ replace Θ and K_i ($i=1,2$) in the scalar inequality (11), respectively. Then we obtain

Theorem 3. *If D is an $\mathcal{M}$-matrix in the sense of Definition 2, then the operator $\mathcal{L}_2: B_q^n\to D_q^n$ in* (7) *is bounded.*

Proof. As D is an $\mathcal{M}$-matrix, the matrix D^{-1} is positive, and we can rewrite (12) as

$$\overline{x} \leq D^{-1}\big(\|x(0)\|_{k_q^n}\overline{e}_1 + \|f\|_{B_q^n}\overline{e}_2\big).$$

Therefore,

$$|\overline{x}| \leq K\big(\|x(0)\|_{k_q^n} + \|f\|_{B_q^n}\big), \tag{13}$$

where $K = \|D^{-1}\| \max\{|e_1|, |e_2|\}$. As $\|x\|_{D_q^n} \leq |\overline{x}|$, we conclude from (13) that $x \in D_q^n$ and $\|x\|_{D_q^n} \leq K(\|b\|_{k_q^n} + \|f\|_{B_q^n})$ for some positive K. Thus, the operator $\mathcal{L}_2 :\ B_q^n \to D_q^n$ is bounded. $\square$

Again, if $q \geq 2$ and one uses the substitution $y(t) = \exp(\lambda t)x(t)$ for some positive λ and Theorems 1, 3 and proves q-stability of the equation for $y(t)$, then this result will imply exponential q-stability of Eq. (1).

The outlined frameworks can be applied to all systems of stochastic differential equations mentioned above as classes (A)–(E). Notice that the second Lyapunov method might be difficult to use in many of these cases.

References

[1] N. V. Azbelev and P. M. Simonov, *Stability of differential Equations with Aftereffect.* Stability and Control: Theory, Methods and Applications, 20. Taylor & Francis, London, 2003.

[2] L. Berezansky, E. Braverman and L. Idels, New global exponential stability criteria for nonlinear delay differential systems with applications to BAM neural networks. *Appl. Math. Comput.* **243** (2014), 899–910.

[3] R. I. Kadiev, Stability of solutions of stochastic functional differential equations. (Russian) *Habilitation thesis, Jekaterinburg*, 2000.

[4] R. I. Kadiev and A. Ponosov, Exponential stability of linear stochastic differential equations with bounded delay and the W-transform. *Electron. J. Qual. Theory Differ. Equ.* **2008**, no. 23, 14 pp.

[5] R. Kadiev and A. Ponosov, Input-to-state stability of linear stochastic functional differential equations. *J. Funct. Spaces* **2016**, Art. ID 8901563, 12 pp.

[6] R. I. Kadiev and A. V. Ponosov, Positive invertibility of matrices and stability of Itô delay differential equations. (Russian) *Differ. Uravn.* **53** (2017), no. 5, 579–590; translation in *Differ. Equ.* **53** (2017), no. 5, 571–582.

[7] R. Sh. Liptser and A. N. Shiryayev, *Theory of Martingales.* Translated from the Russian by K. Dzjaparidze [Kacha Dzhaparidze]. Mathematics and its Applications (Soviet Series), 49. Kluwer Academic Publishers Group, Dordrecht, 1989.