Representation of the Solution of the Inhomogeneous Wave Equation in a Half-Strip in the Form of Finite Sum of Addends, Depending on Boundary, Initial Values of the Solution and Right-Hand Side of the Equation

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In a plane of independent variables x and t in the half-strip $D_\infty:\ 0<x<l,\ t>0$ consider the mixed problem of finding solution $u(x,t)$ of the linear inhomogeneous wave equation of the form

$$\square u=f(x,t),\quad (x,t)\in D_\infty, \tag{1}$$

satisfying the following initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\le x\le l, \tag{2}$$

and boundary conditions

$$u(0,t)=\mu_1(t),\ \ t\ge 0, \tag{3}$$
$$u(l,t)=\mu_2(t),\ \ t\ge 0, \tag{4}$$

where f, φ, ψ, μ_i, $i=1,2$, are given functions and u is unknown real function, and $\square:=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}$.

It is easy to see that for

$$f\in C^1(\overline{D}_\infty),\ \ \varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \mu_i\in C^2([0,\infty)),\ \ i=1,2,$$

the necessary conditions for solvability of problem (1)–(4) in the class $C^2(\overline{D}_\infty)$ are the following second order agreement conditions

$$\begin{gathered}\varphi(0)=\mu_1(0),\ \ \psi(0)=\mu_1'(0),\ \ \mu_1''(0)-\varphi''(0)=f(0,0),\\ \varphi(l)=\mu_2(0),\ \ \psi(l)=\mu_2'(0),\ \ \mu_2''(0)-\varphi''(l)=f(l,0).\end{gathered}$$

Let

$$m=m(t):=\Big[\frac{t}{l}\Big],\ \ t>0,$$

where $[\cdot]$ is an integer part of a real number.

Let us divide the domain $E_m:\ 0<x<l,\ ml<t<(m+1)l,\ m=0,1,2,\dots$, which is a quadrat with vertices in points $A_m(0,ml)$, $B_m(0,(m+1)l)$, $C_m(l,(m+1)l)$ and $D_m(l,ml)$ into four rectangular triangles: $E_m^1:=\Delta A_mO_mD_m$, $E_m^2:=\Delta A_mO_mB_m$, $E_m^3:=\Delta D_mO_mC_m$ and $E_m^4:=\Delta B_mO_mC_m$, where point $O_m(\frac{l}{2},(m+\frac{1}{2})l)$ is a center of the quadrat E_m.

Below we get the representation of the classical solution $u\in C^2(\overline{D}_\infty)$ of problem (1)–(4) in the half-strip D_∞ in the form of finite sum of addends, depending on boundary, initial values of this solution and right-hand side of equation (1).

First let $P = P(x,t) \in E_0$. In the triangle E_0^1 due to (2) and the d'Alembert's formula, the equality [7, p. 59]

$$
\begin{aligned}
u(x,t) &= A_1(\varphi,\psi,f)(x,t)\\
&:= \frac{1}{2}\big[\varphi(x-t)+\varphi(x+t)\big] + \frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^1} f(\xi,\tau)\,d\xi\,d\tau, \quad (x,t)\in E_0^1 \qquad (5)
\end{aligned}
$$

is valid, where $\Omega_{x,t}^1$ is the triangle with vertices at the points (x,t), $(x-t,0)$ and $(x+t,0)$.

As it is known, for any twice continuously differentiable function v and characteristic to equation (1) rectangle $PP_1P_2P_3$ from its domain of definition the equality [1, p. 173]

$$
v(P) = v(P_1) + v(P_2) - v(P_3) + \frac{1}{2}\int\limits_{PP_1P_2P_3} \Box\, v(\xi,\tau)\,d\xi\,d\tau \qquad (6)
$$

is valid, where P and P_3, P_1 and P_2 are opposite vertices of this rectangle, and the ordinate of point P is larger than those of the rest points.

Let now $P \in E_0^2$. Then, using equality (6) for characteristic rectangle with vertices at the points $P(x,t)$, $P_1(0,t-x)$, $P_2(t,x)$ and $P_3(t-x,0)$, and formula (5) for point $P_2(t,x) \in E_0^1$, in view of (1) and (3) we have

$$
\begin{aligned}
u(x,t) &= A_2(\varphi,\psi,\mu_1,f)(x,t)\\
&:= \mu_1(t-x) + \frac{1}{2}\big[\varphi(t+x)-\varphi(t-x)\big] + \frac{1}{2}\int\limits_{t-x}^{t+x}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^2} f(\xi,\tau)\,d\xi\,d\tau, \quad (x,t)\in E_0^2. \qquad (7)
\end{aligned}
$$

Here $\Omega_{x,t}^2$ is the quadrangle $PP_2^*P_3P_1$, where $P_2^* := (t+x,0)$.

Analogously, we have

$$
\begin{aligned}
u(x,t) &= A_3(\varphi,\psi,\mu_2,f)(x,t) := \mu_2(x+t-l)\\
&+ \frac{1}{2}\big[\varphi(x-t)-\varphi(2l-x-t)\big] + \frac{1}{2}\int\limits_{x-t}^{2l-x-t}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^3} f(\xi,\tau)\,d\xi\,d\tau, \quad (x,t)\in E_0^3, \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
u(x,t) &= A_4(\varphi,\psi,\mu_1,\mu_2,f)(x,t) := \mu_1(t-x) + \mu_2(x+t-l)\\
&- \frac{1}{2}\big[\varphi(t-x)+\varphi(2l-t-x)\big] + \frac{1}{2}\int\limits_{t-x}^{2l-t-x}\psi(\tau)\,d\tau + \frac{1}{2}\int\limits_{\Omega_{x,t}^4} f(\xi,\tau)\,d\xi\,d\tau, \quad (x,t)\in E_0^4. \qquad (9)
\end{aligned}
$$

Here $\Omega_{x,t}^3$ is a quadrangle with vertices $P^3(x,t)$, $P_1^3(l,x+t-l)$, $P_2^3(x-t,0)$ and $P_3^3(2l-x-t,0)$, while $\Omega_{x,t}^4$ is a pentagon with vertices $P^4(x,t)$, $P_1^4(0,t-x)$, $P_2^4(t-x,0)$, $P_3^4(2l-x-t,0)$ and $P_4^4(l,x+t-l)$.

If the point $P_0 := P_0(x,t) \in E_m$, $m \geq 1$, then denote by $P_0M_1P_1N_1$ the characteristic rectangle with respect to equation (1), whose vertices M_1 and N_1 lay on the straight lines $x=0$ and $x=l$, respectively, i.e. $M_1 := (0,t-x)$, $N_1 := (l,t+x-l)$, $P_1 := (l-x,t-l)$. Since $P_1 \in E_{m-1}$, then by analogy we can consider the characteristic rectangle $P_1M_2P_2N_2$, whose vertices M_2 and N_2 lay on the straight lines $x=0$ and $x=l$, respectively. Continuing this process we get the characteristic rectangle $P_{i-1}M_iP_iN_i$ with vertices M_i and N_i, respectively, on the straight lines $x=0$ and $x=l$, and due to $P_0\in E_m$,

$$P_m\in E_0, \tag{10}$$

where $P_m=(l-x,t-ml)$ if m is odd, and $P_m=(x,t-ml)$ if m is even. At the same time if the point $P_0\in E^1_m(E^4_m)$, then $P_m\in E^1_0(E^4_0)$ for any m, and if $P_0\in E^2_m(E^3_m)$, then $P_m\in E^3_0(E^2_0)$ for odd m and $P_m\in E^2_0(E^3_0)$ for even m. For the coordinates of the points M_i and N_i we have

$$\begin{gathered}M_i=(0,t-x-(i-1)l),\ \ N_i=(l,t+x-il),\ \ i=1,3,5,\dots,\\ M_i=(0,t+x-il),\ \ N_i=(l,t-x-(i-1)l),\ \ i=2,4,6,\dots\end{gathered}$$

By induction over number m it can be proved the validity of the following representation of the solution $u\in C^2(\overline{D}_\infty)$ of problem (1)–(4) in the half-strip D_∞

$$u(P_0)=\sum_{i=1}^m(-1)^{i-1}\bigg[\mu_1(M_i)+\mu_2(N_i)+\frac12\int\limits_{P_{i-1}M_iP_iN_i}f(\xi,\tau)\,d\xi\,d\tau\bigg]+(-1)^mu(P_m),\ \ P_0\in E_m,$$

where due to (10) in the case of odd m

$$u(P_m)=\begin{cases}A_1(\varphi,\psi,f)(P_m), & P_0\in E^1_m,\\ A_3(\varphi,\psi,\mu_2,f)(P_m), & P_0\in E^2_m,\\ A_2(\varphi,\psi,\mu_1,f)(P_m), & P_0\in E^3_m,\\ A_4(\varphi,\psi,\mu_1,\mu_2,f)(P_m), & P_0\in E^4_m,\end{cases}$$

while for even m

$$u(P_m)=\begin{cases}A_1(\varphi,\psi,f)(P_m), & P_0\in E^1_m,\\ A_2(\varphi,\psi,\mu_1,f)(P_m), & P_0\in E^2_m,\\ A_3(\varphi,\psi,\mu_2,f)(P_m), & P_0\in E^3_m,\\ A_4(\varphi,\psi,\mu_1,\mu_2,f)(P_m), & P_0\in E^4_m.\end{cases}$$

Here the operators A_i, $i=1,2,3,4$ are defined by formulas (5), (7)–(9).

The obtained representation will unconditionally find application during a study of other initial-boundary value problems both for linear and nonlinear hyperbolic equations and systems. Let us note that other representations of the solution of problem (1)–(4) in the form of infinite series are given in [1–9].

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