Description by Suslin's Sets of Bounded Families of Liapunov's Characteristic Exponents in the Full Perron's Effect of Their Value Change

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We consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^2, \quad t \geq 0, \tag{1}$$

with a bounded continuously differentiable matrix of coefficients $A(t)$ and with negative characteristic exponents $\lambda_1(A) \leq \lambda_2(A) < 0$. The system is a linear approximation for the nonlinear system

$$\dot{y} = A(t)y + f(t,y), \quad y = (y_1, y_2)^\top \in \mathbb{R}^2, \quad t \geq 0. \tag{2}$$

In addition, the so-called m-perturbation $f(t,y)$ is continuously differentiable in its arguments $t \geq 0$ and $y_1, y_2 \in \mathbb{R}$ and has an $m \geq 1$ order of smallness in some neighbourhood of the origin and growth outside of it:

$$\|f(t,y)\| \leq C_f \|y\|^m, \quad m > 1, \quad y \in \mathbb{R}^2, \quad t \geq 0. \tag{3}$$

Perron's effect [7], [6, pp. 50, 51] of sign and value change of characteristic exponents establishes the existence of system (1) with negative Lyapunov exponents and 2-perturbation (3) such that all nontrivial solutions of the perturbed system (2) turn out to be infinitely continuable and have finite Lyapunov exponents equal to:

(1) the negative higher exponent λ_2 of the initial system (1) for solutions starting at the initial moment on the axis $y_1 = 0$ (that allows one to consider Perron's effect not full);

(2) a certain positive value for all the rest solutions (calculated in [2, pp. 13–15]).

A number of works written by the author and jointly with Korovin contain various versions of the full Perron's effect when all nontrivial solutions of the nonlinear system (2) with m-perturbation (3) are infinitely continuable (this is not the case in a general case) and have finite positive Lyapunov exponents under negative exponents of the system of linear approximation (1). These versions correspond to different types of the set $\lambda(A,f) \subset (0,+\infty)$ of Lyapunov's characteristic exponents of all nontrivial solutions of the perturbed system (2), to distribution of these solutions with respect to the exponents from the set $\lambda(A,f)$ and, finally, to an arbitrary order of systems (1) and (2).

In particular, it is stated in [3, 4] that the sets $\lambda(A,f)$ in this full Perron's effect are Suslin's ones [1, pp. 97, 98, 192]. For a complete description of (bounded) families $\lambda(A,f) \subset (0,+\infty)$ in that effect there arises an inverse question on the realization of an arbitrary bounded Suslin's set $S \subset (0, +\infty)$ by the family $\lambda(A, f)$ of characteristic exponents of a certain perturbed system (2), i.e., the question on the realization of the equality $\Lambda(A_s, f_s) \equiv S$ for the above-mentioned matrix $A_s(t)$ and vector-function $f_s(t, y)$.

The positive and stronger answer to the above question in classes of infinitely differentiable matrices $A(t)$ and vector-functions $f(t, y)$ in the corresponding spaces (that will be additionally supposed in the sequel) is contained in the present report.

The following theorem is valid.

Theorem 1 ([5]). *For arbitrary parameters $m > 1$, $\lambda_1 \le \lambda_2 < 0$ and arbitrary bounded on the axis $\mathbb{R}_0 = \mathbb{R} \setminus 0$ Baer's 1st class functions*

$$\psi_i : \mathbb{R}_0 \to [\beta_i, b_i] \subset (0, +\infty), \ \ b_1 \le \beta_2, \ \ i = 1, 2,$$

there exist a linear system (1) *with bounded infinitely differentiable on the semi-axis* $[t_0, +\infty)$ *coefficients and exponents $\lambda_1(A) = \lambda_1 \le \lambda_2 = \lambda_2(A)$ and the infinitely differentiable in its arguments $t \ge t_0$ and $y_1, y_2 \in \mathbb{R}$ m-perturbation $f(t, y)$ such that all nontrivial solutions $t(t, c)$ of the nonlinear system* (2) *are infinitely continuable to the right and have characteristic exponents*

$$\lambda[y(\,\cdot\,, c)] = \begin{cases} \psi_1(c_1), & c_1 \ne 0, \ \ c_2 = 0, \\ \psi_2(c_2), & c_2 \ne 0, \ ; \ c = (c_1, c_2) \in \mathbb{R}^2. \end{cases}$$

The above theorem results in the following corollary.

Corollary 1 ([5]). *For arbitrary parameters $m > 1$, $\lambda_1 \le \lambda_2 < 0$ and the bounded Suslin's set $S \subset (0, +\infty)$ there exist systems* (1) *and* (2) *mentioned in the above theorem such that the set of characteristic exponents of nontrivial solutions of the latter coincides with the set S.*

When proving the theorem we have used the following statements.

Lemma 1 ([5]). *Let the bounded on the axis $R_0 = R \setminus \{0\}$ function*

$$\psi : R_0 \to |\beta_0, b_0|, \ \ -\infty < \beta_0 < b_0 < +\infty,$$

be Baer's 1st class function. Then for arbitrary constants $\beta < \beta_0$ and $b > b_0$ there exists a sequence $\{\psi_n(x)\}$ of infinitely differentiable uniformly bounded on the axis $\mathbb{R}_0$ functions $\psi_n : R_0 \Longrightarrow [\beta, b]$, $n \in \mathbb{N}$, converging on that axis to the function $\psi(x)$.

Lemma 2 ([5]). *For arbitrary numbers $\varepsilon > 0$ and continuous on the axis $\mathbb{R}_0$ function $F_0 : \mathbb{R}_0 \to \mathbb{R}$ there exists an infinitely differentiable on that axis function $F : \mathbb{R}_0 \to \mathbb{R}$ for which the inequality*

$$|F(x) - F_0(x)| \le \varepsilon, \ \ x \in \mathbb{R}_0$$

is fulfilled.

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