On the Detection of Exact Number of Limit Cycles for Autonomous Systems on the Cylinder

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Consider the planar autonomous differential system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where the functions $P, Q : R^2 \to R$ are 2π-periodic in the first variable. Under this assumption we can identify the phase space of (1) with the cylinder $Z := S^1 \times R$, where S^1 is the unit circle. The most difficult problem in the qualitative investigation of autonomous differential systems is the localization and the estimate of the number of limit cycles.

In the case of a cylindrical phase space we have to distinguish two kinds of limit cycles. A limit cycle of system (1) which does not surround Z is called a limit cycle of the first kind, otherwise it is called a limit cycle of the second kind. Whereas the existence of a limit cycle of the first kind of system (1) requires the existence of an equilibrium point, a limit cycle of the second kind can exist without the existence of any equilibrium point [1, p. 34–35], [2, p. 218–227]. For the study of limit cycles of the first kind, the methods for planar autonomous systems can be applied (see, e.g. [2]). In particular, a well-known way to get an upper bound for the number of limit cycles of the first kind in planar systems is to check whether the criteria of I. Bendixson and H. Dulac [2] can be applied.

The method of the Dulac function has been extended by L. Cherkas [3]. The type of functions he has introduced nowadays is called Dulac–Cherkas function [7]. The existence of a Dulac–Cherkas function has the following advantages over a Dulac function: it guarantees that all limit cycles are hyperbolic (there is no multiple limit cycle), it provides some annuli containing a unique limit cycle (approximate localization of a limit cycle), it yields a simple criterion to determine the stability of limit cycles and provides lower and upper bounds for their maximum number. These functions have been applied by L. Cherkas and his coauthors also for the investigation of limit cycles of the second kind [4, 5, 8].

The fundamental importance of a Dulac–Cherkas function consists in the fact that its zero-level set defines curves which are crossed transversally by the trajectories of the corresponding system. We denote these curves in what follows as transversal curves. By this way, the cylindrical phase space is divided into doubly connected regions, where we have to distinguish between interior regions whose boundaries consist of transversal curves and which contain a unique limit cycle, and two outer regions, where only one boundary of these regions is a transversal curve and which contain at most one limit cycle. To be able to determine the exact number of limit cycles we have to investigate the existence of a limit cycle in the two outer regions. The main contribution of this paper is to show that the existence of a unique limit cycle in the outer regions can be established either by means of the existence of additional Dulac–Cherkas functions or by factorized Dulac functions. Thus, we present results on the exact number of limit cycles of the second kind.

The estimate of the number of limit cycles in some given region depends also on the structure of the region itself. Hence, our first assumption reads

(A_0). Let G be an open bounded doubly connected region on Z whose boundary consists of two simple closed curves Δ_u and Δ_l surrounding Z. We suppose that Δ_u is located above Δ_l, that is, Δ_u is the upper boundary and Δ_l is the lower boundary of G.

We denote by $C^1_{2\pi}(G,R)$ the space of continuously differentiable functions mapping G into R and which are 2π-periodic in the first variable. For the following we assume:

(A_1). The functions P and Q belong to the space $C^1_{2\pi}(G,R)$.

(A_2). G does not contain an equilibrium point of (1).

Assumption (A_2) implies that any closed orbit of system (1) completely located in G must surround the cylinder Z. That means that any limit cycle of system (1) in G is a limit cycle of the second kind which we denote by Γ. Our goal is to determine or at least to estimate the number of limit cycles of the second kind of system (1) in G. We denote this number by $\sharp\,\Gamma(G)$. The vector field defined by system (1) is denoted by X.

A known tool to estimate the number $\sharp\,\Gamma(G)$ is the Dulac function.

Definition 1. A function $D \in C^1_{2\pi}(G,R)$ is called a Dulac function of system (1) in G if $\operatorname{div}(DX)$ does not change sign in G.

The following result is well-known [2].

Theorem 1. *Suppose the assumptions (A_0)–(A_2) are satisfied. If there is a Dulac function of system* (1) *in the region G, then it holds $\sharp\,\Gamma(G) \leqslant 1$.*

The concept of the Dulac function has been generalized by L. Cherkas [3]. For this new class of functions we introduced in [7] the name Dulac–Cherkas function.

Definition 2. Suppose the assumptions (A_0) and (A_1) are satisfied. A function $\Psi \in C^1_{2\pi}(G,R)$ is called a Dulac–Cherkas function of system (1) in G if the set $W := \{(x,y) \in G:\ \Psi(x,y) = 0\}$ does not contain a curve which is a trajectory of system (1) and there is a real number $k \neq 0$ such that the following condition holds

$$\Phi(x,y,k) := (\operatorname{grad}\Psi, X) + k\Psi \operatorname{div} X \geqslant 0\ (\leqslant 0)\ \ \forall\,(x,y) \in G, \tag{2}$$

where the set $V_k := \{(x,y) \in G:\ \Phi(x,y,k) = 0\}$ has measure zero.

For $k = 1$ the definition of a Dulac–Cherkas function coincides with the definition of a Dulac function. If Ψ is a Dulac–Cherkas function of system (1) in G, then $|\Psi|^{1/k}$ is a Dulac function of (1) in $G \setminus W$. For the following results we introduce the assumption.

(A_3). There is a Dulac–Cherkas function Ψ of system (1) in G with $k < 0$ such that the set W consists of $l \geqslant 1$ simple closed curves $w_1,\dots,w_l$ surrounding the cylinder Z (we call them ovals) and which do not meet each other as well as the boundaries Δ_u and Δ_l of G.

Remark 1. If we consider the function Φ on any oval w_i of the set W, then we get from (2)

$$\Phi(x,y,k)_{|w_i} = (\operatorname{grad}\Psi, X)_{|w_i} = \frac{d\Psi}{dt}_{|w_i} \geqslant 0\ (\leqslant 0),$$

where d/dt denotes the differentiation along system (1). The conditions in Definition 2 implies

$$\frac{d\Psi}{dt}_{|w_i} \not\equiv 0,$$

and we can conclude that any trajectory of (1) which meets any oval w_i will cross it for increasing or decreasing t.

Concerning the location of these ovals on the cylinder Z we assume that the oval w_i is located over the oval w_{i+1}. The doubly connected subregion of G bounded by w_i and w_{i+1} is denoted by Z_i, $i = 1, \dots, l-1$, the region bounded by Δ_u and w_1 is denoted by Z_0, and the region bounded by w_l and Δ_l is denoted by Z_l, which are the outer regions.

The following result is also known [5].

Theorem 2. *Suppose that the assumptions (A_0)–(A_3) are valid. Then it holds:*

(i) *Each region Z_i, $1 \leqslant i \leqslant l-1$, contains a unique limit cycle Γ_i of the second kind of system* (1)*. Γ_i is hyperbolic, it is stable (unstable) if $\Phi(x,y,k)\Psi(x,y) > 0$ (< 0) in Z_i.*

(ii) *The regions Z_0 and Z_l may contain a unique limit cycle of the second kind which is hyperbolic, and therefore, it implies immediately the estimate*

$$l - 1 \leqslant \sharp\,\Gamma(G) \leqslant l + 1. \tag{3}$$

Remark 2. Under the assumptions (A_0)–(A_3) any improvement of estimate (3) is connected with the existence or absence of a limit cycle of the second kind in the regions Z_0 and Z_l.

Now we want to establish conditions for the existence of a limit cycle of the second kind in Z_0 and/or in Z_l. By Remark 1 we can conclude that any trajectory of system (1) that meets an oval w_i of the set W will cross w_i for increasing or decreasing t. Therefore, appropriate Dulac–Cherkas functions can be used to construct doubly-connected regions to which the Poincaré–Bendixson theorem can be applied.

Theorem 3. *Suppose that the assumptions (A_0)–(A_3) are valid. Additionally, we assume the existence of a second Dulac–Cherkas function Ψ_0 of system* (1) *in some doubly connected subregion $\widetilde{Z}_0$ of Z_0 whose boundaries surround Z such that the corresponding set $W_0 := \{(x,y) \in \widetilde{Z}_0 :\ \Psi_0(x,y) = 0\}$ consists of exactly one oval v_0 and where the ovals v_0 and w_1 form the boundaries of the doubly connected region Z_{00} to which the Poincaré-Bendixson theorem can be applied. Then it holds*

$$l \leqslant \sharp\,\Gamma(G) \leqslant l + 1.$$

In the same way we can formulate the similar theorem for the region Z_l.

Remark 3. If the assumptions of Theorem 3 are fulfilled simultaneously for both regions Z_0 and Z_l, then it holds

$$\sharp\,\Gamma(G) = l + 1. \tag{4}$$

The exact number of limit cycles of the second kind in G can be also determined by means of an additional Dulac–Cherkas function defined in the same region G.

Theorem 4. *Suppose the assumptions (A_0)–(A_3) are valid. Additionally, we assume the existence of a second Dulac–Cherkas function Ψ_1 of system* (1) *in G with $k_1 < 0$ such that the corresponding set W_1 consists of $l+2$ ovals. Then estimate* (4) *holds.*

As the next step we present another approach based on factorized Dulac functions.

Let χ_1 and χ_2 be functions of the space $C^1_{2\pi}(G,R)$. For the following, we introduce the sets

$$U_i := \big\{(x,y) \in G :\ \chi_i(x,y) = 0\big\},\ \ i = 1, 2.$$

We denote by U the set $U := U_1 \cup U_2$ and define the function $D : G \setminus U \to R^+$ by

$$D(x,y,k_1,k_2) := |\chi_1(x,y)|^{k_1} |\chi_2(x,y)|^{k_2}, \tag{5}$$

where k_1 and k_2 are real numbers. For the divergence of the vector field we get from (5) in the region $G \setminus U$

$$\operatorname{div}(DX) = |\chi_1|^{k_1-1}|\chi_2|^{k_2-1} \operatorname{sgn}\chi_1 \operatorname{sgn}\chi_2 \big(\chi_1\chi_2 \operatorname{div} X + k_1\chi_2(\operatorname{grad}\chi_1, X) + k_2\chi_1(\operatorname{grad}\chi_2, X)\big).$$

Our goal is to derive conditions such that D is a Dulac function in some region of $G \setminus U$. Therefore, additionally we suppose

(C_1). There are functions $\chi_1, \chi_2 \in C^1_{2\pi}(G, R)$ and real numbers k_1, k_2 such that in G the following condition holds

$$\Theta(x, y, k_1, k_2) := \chi_1\chi_2 \operatorname{div} X + k_1\chi_2(\operatorname{grad}\chi_1, X) + k_2\chi_1(\operatorname{grad}\chi_2, X) < 0 \ (> 0).$$

Since we are interested in estimating the number of limit cycles of the second kind in G, we assume

(C_2). The set U consists in G of n ovals surrounding Z.

We denote by $v_1, \dots, v_m$ the ovals of U, where v_i is located above v_{i+1}. We denote by Z_i, $1 \leqslant i \leqslant n-1$, the open doubly connected region bounded by v_i and v_{i+1}, Z_0 is the open doubly connected region bounded by Δ_u and v_1, Z_n is the open doubly connected region bounded by v_n and Δ_l.

Theorem 5. *Suppose the assumptions* (A_0), (A_1), (A_2) *and* (C_1) *with* $k_1 < 0, k_2 < 0$, *and* (C_2) *are valid. Then it holds:*

(i) *Each region* Z_i, $1 \leqslant i \leqslant n-1$, *contains a unique limit cycle* Γ_i *of the second kind of system* (1). Γ_i *is hyperbolic and stable (unstable) if the inequality*

$$\frac{\Theta(x, y, k_1, k_2)}{\chi_1(x, y)\chi_2(x, y)} < 0 \ (> 0)$$

is valid in Z_i.

(ii) *In each of the regions* Z_0 *and* Z_n *a unique hyperbolic limit cycle of the second kind could be located.*

A detailed presentation of our approaches to check the existence of a limit cycle in the regions Z_0 and Z_l or Z_n by means of an additional Dulac–Cherkas functions or by special factorized Dulac functions and their application to some classes of systems (1) are contained in our paper [6].

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