On Below Estimates for the First Eigenvalue of a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all measurable locally integrable on $(0,1)$ functions with non-negative values such that the following integral condition hold

$$\int_0^1 x^{\alpha}(1-x)^{\beta} Q^{\gamma}(x)\,dx = 1, \quad \alpha,\beta,\gamma \in \mathbb{R}, \quad \gamma \neq 0, \tag{1.3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty. \tag{1.4}$$

A function y is a *solution to problem* (1.1), eqrefTelnova eq 2 if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

For $\gamma < 0$, $\alpha \le 2\gamma - 1$, $-\infty < \beta < +\infty$ or $\gamma < 0$, $\beta \le 2\gamma - 1$, $-\infty < \alpha < +\infty$, the set $T_{\alpha,\beta,\gamma}$ is empty, the first eigenvalue of problem (1.1), (1.2) does not exist. Given $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ or $\gamma > 0$, $-\infty < \alpha$, $\beta < +\infty$, $Q \in T_{\alpha,\beta,\gamma}$, we obtain

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y], \quad \text{where} \quad R[Q,y] = \frac{\int\limits_0^1 y'^2\,dx - \int\limits_0^1 Q(x)y^2\,dx}{\int\limits_0^1 y^2\,dx}.$$

For any α, β, γ, $\gamma \neq 0$, for any $Q \in T_{\alpha,\beta,\gamma}$, the following relations hold

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \inf_{y \in H_0^1(0,1)} R[Q,y] \le \inf_{y \in H_0^1(0,1)\setminus\{0\}} \frac{\int\limits_0^1 y'^2\,dx}{\int\limits_0^1 y^2\,dx} = \pi^2.$$

2 Main results

Theorem 2.1.

1. *If* $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$ *or* $0 < \gamma < 1$, *then* $m_{\alpha,\beta,\gamma} = -\infty$.

2.1. *If* $\gamma = 1$, $\alpha, \beta \leqslant 0$, *then* $m_{\alpha,\beta,\gamma} \geqslant \frac{3}{4}\pi^2$.

2.2. *If* $\gamma = 1$, $\beta \leqslant 0 < \alpha \leqslant 1$ *or* $\alpha \leqslant 0 < \beta \leqslant 1$, *then* $m_{\alpha,\beta,\gamma} \geqslant 0$.

2.3. *If* $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$, *then* $m_{\alpha,\beta,\gamma} \geqslant 0$.

2.4. *If* $\gamma > 1$, $\alpha, \beta \leqslant 0$, *then* $m_{\alpha,\beta,\gamma} \geqslant 0$.

Proof. By the Hölder inequality, for any $y \in H_0^1(0,1)$, for any $x \in (0,1)$, we have

$$y^2(x) = \Big(\int_0^x y'(t)\,dt\Big)^2 \leqslant x \int_0^x {y'}^2(t)\,dt, \tag{2.1}$$

$$y^2(x) = \Big(-\int_x^1 y'(t)\,dt\Big)^2 \leqslant (1-x) \int_x^1 {y'}^2(t)\,dt.$$

Then

$$\frac{y^2}{x(1-x)} = \frac{y^2}{x} + \frac{y^2}{1-x} \leqslant \int_0^x {y'}^2(t)\,dt + \int_x^1 {y'}^2(t)\,dt = \int_0^1 {y'}^2(t)\,dt,$$

$$y^2(x) \leqslant x(1-x) \int_0^1 {y'}^2(t)\,dt. \tag{2.2}$$

1.1. If $\gamma < 0$, $\alpha, \beta > 2\gamma - 1$, then there exists a number $r > 0$ such that $\alpha > 2\gamma - 1 + r$, $\beta > 2\gamma - 1 + r$. For $0 < \varepsilon < 1$, consider the function $Q_\varepsilon \in T_{\alpha,\beta,\gamma}$:

$$Q_\varepsilon(x) = \begin{cases} r^{\frac{1}{\gamma}}(1-\varepsilon)^{\frac{1}{\gamma}}\varepsilon^{-\frac{r}{\gamma}}x^{-\frac{\alpha+1-r}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & 0 < x \leqslant \varepsilon; \\ r^{\frac{1}{\gamma}}(1-\varepsilon)^{-\frac{r}{\gamma}}\varepsilon^{\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta+1-r}{\gamma}}, & \varepsilon < x < 1. \end{cases}$$

By the Hölder inequality, for any function $y \in H_0^1(0,1)$, we have

$$\int_0^1 y^2\,dx = \int_0^\varepsilon y^2\,dx + \int_\varepsilon^1 y^2\,dx \leqslant \frac{\varepsilon^2}{2}\int_0^\varepsilon {y'}^2\,dx + r^{-\frac{1}{\gamma}}\varepsilon^{-\frac{1}{\gamma}}(1-\varepsilon)^{\frac{r}{\gamma}}\int_\varepsilon^1 Q_\varepsilon(x)y^2\,dx.$$

Then

$$\int_0^1 Q_\varepsilon(x)y^2\,dx \geqslant \int_\varepsilon^1 Q_\varepsilon(x)y^2\,dx \geqslant r^{\frac{1}{\gamma}}\varepsilon^{\frac{1}{\gamma}}(1-\varepsilon)^{-\frac{r}{\gamma}}\Big(\int_0^1 y^2\,dx - \frac{\varepsilon^2}{2}\int_0^1 {y'}^2\,dx\Big).$$

For any function $y_* \in H_0^1(0,1)$, for example, for $y_* = \sin \pi x$,

$$R[Q_\varepsilon, y_*] \leqslant \frac{\int_0^1 {y_*'}^2\,dx + r^{\frac{1}{\gamma}}\varepsilon^{\frac{1}{\gamma}}(1-\varepsilon)^{-\frac{r}{\gamma}}\big(\frac{\varepsilon^2}{2}\int_0^1 {y_*'}^2\,dx - \int_0^1 {y_*}^2\,dx\big)}{\int_0^1 {y_*}^2\,dx}.$$

Therefore,

$$\inf_{Q\in T_{\alpha,\beta,\gamma}}\ \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y]\leqslant \lim_{\varepsilon\to 0}\ \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q_\varepsilon,y]\leqslant \lim_{\varepsilon\to 0} R[Q_\varepsilon,y_*]=-\infty.$$

1.2. Let $0<\gamma<1$ and α,β be arbitrary real numbers. For $0<\varepsilon<1$, consider the function $Q_\varepsilon\in T_{\alpha,\beta,\gamma}$:

$$Q_\varepsilon(x)=\begin{cases} 0, & 0\leqslant x<\dfrac{1}{2}-\dfrac{\varepsilon}{2},\ \ \dfrac{1}{2}+\dfrac{\varepsilon}{2}<x\leqslant 1;\\[2mm] \varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}, & \dfrac{1}{2}-\dfrac{\varepsilon}{2}\leqslant x\leqslant \dfrac{1}{2}+\dfrac{\varepsilon}{2}\,. \end{cases}$$

If $y_*=\sin\pi x$ and $C_\varepsilon=\min_{[\frac{1}{2}-\frac{\varepsilon}{2},\frac{1}{2}+\frac{\varepsilon}{2}]} x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}$, then

$$\int_0^1 Q_\varepsilon(x){y_*}^2\,dx=\int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}} \varepsilon^{-\frac{1}{\gamma}}x^{-\frac{\alpha}{\gamma}}(1-x)^{-\frac{\beta}{\gamma}}\sin^2\pi x\,dx$$

$$\geqslant C_\varepsilon\cdot\varepsilon^{-\frac{1}{\gamma}}\int_{\frac{1}{2}-\frac{\varepsilon}{2}}^{\frac{1}{2}+\frac{\varepsilon}{2}}\frac{1-\cos 2\pi x}{2}\,dx=C_\varepsilon\cdot\varepsilon^{-\frac{1}{\gamma}}\Big(\frac{\varepsilon}{2}+\frac{\sin\pi\varepsilon}{2\pi}\Big).$$

Similarly to case 1.1, we obtain $m_{\alpha,\beta,\gamma}=-\infty$.

2.1. Let $\gamma=1$ and $\alpha,\beta\leqslant 0$. It is known (see, for ex., [1]) that for any $y\in H_0^1(0,1)$, the inequality

$$\sup_{[0,1]} y^2\leqslant \frac{1}{4}\int_0^1 {y'}^2\,dx$$

holds. For any functions $Q\in T_{\alpha,\beta,\gamma}$ and $y\in H_0^1(0,1)$, we obtain

$$\int_0^1 Q(x)y^2\,dx\leqslant \sup_{[0,1]} y^2\int_0^1 Q(x)x^\alpha(1-x)^\beta\,dx\leqslant \sup_{[0,1]} y^2\leqslant \frac{1}{4}\int_0^1 {y'}^2\,dx.$$

Therefore,

$$m_{\alpha,\beta,\gamma}\geqslant \frac{3}{4}\inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 {y'}^2\,dx}{\int_0^1 y^2\,dx}=\frac{3}{4}\,\pi^2.$$

2.2. Let $\gamma=1$, $\beta\leqslant 0<\alpha\leqslant 1$. In virtue of (2.1), for any function $Q\in T_{\alpha,\beta,\gamma}$, we have

$$\int_0^1 Q(x)y^2\,dx\leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha}\int_0^1 Q(x)x^\alpha(1-x)^\beta\,dx\leqslant \sup_{[0,1]}\frac{y^2}{x}\leqslant \int_0^1 {y'}^2\,dx.$$

Then

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\ \inf_{y\in H_0^1(0,1)\setminus\{0\}}\frac{\int_0^1 {y'}^2-\int_0^1 Q(x)y^2\,dx}{\int_0^1 y^2\,dx}\geqslant 0.$$

The case $\alpha \leqslant 0 < \beta \leqslant 1 = \gamma$ is symmetrical to the case $\beta \leqslant 0 < \alpha \leqslant 1 = \gamma$.

2.3. Let $\gamma = 1$, $0 < \alpha, \beta \leqslant 1$. In virtue of (2.2),

$$\int\limits_0^1 Q(x)y^2\,dx \leqslant \sup_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta} \int\limits_0^1 Q(x)x^\alpha(1-x)^\beta\,dx \leqslant \sup_{[0,1]} \frac{y^2}{x(1-x)} \leqslant \int\limits_0^1 y'^2\,dx.$$

and also $m_{\alpha,\beta,\gamma} \geqslant 0$.

2.4. Let $\gamma > 1$, $\alpha, \beta \leqslant 0$. By the Hölder inequality, for any $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_0^1(0,1)$, we obtain the same result due to

$$\int\limits_0^1 Q(x)y^2\,dx \leqslant \Big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \Big)^{\frac{\gamma-1}{\gamma}} \leqslant \Big(\int\limits_0^1 |y|^{\frac{2\gamma}{\gamma-1}}\,dx \Big)^{\frac{\gamma-1}{\gamma}} \leqslant \int\limits_0^1 y'^2\,dx. \tag{2.3}$$

3 On precise estimates for $m_{\alpha,\beta,\gamma}$ as $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$

Theorem 3.1. *If $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$, then there exist functions $Q_* \in T_{\alpha,\beta,\gamma}$ and $u \in H_0^1(0,1)$, $u > 0$ on $(0,1)$, such that $m_{\alpha,\beta,\gamma} = R[Q_*, u]$, moreover, u satisfies equation*

$$u'' + mu = -x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{\gamma+1}{\gamma-1}} \tag{3.1}$$

and the integral condition

$$\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2\gamma}{\gamma-1}}\,dx = 1. \tag{3.2}$$

Proof. Let $\gamma > 1$, $\alpha, \beta < 2\gamma - 1$. In virtue of (2.3), for any $Q \in T_{\alpha,\beta,\gamma}$ and $y \in H_0^1(0,1)$,

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y] \geqslant \inf_{y \in H_0^1(0,1)\setminus\{0\}} G[y] = m,$$

where

$$G[y] = \frac{\int\limits_0^1 y'^2\,dx - \big(\int\limits_0^1 x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}|y|^{\frac{2\gamma}{\gamma-1}}\,dx \big)^{\frac{\gamma-1}{\gamma}}}{\int\limits_0^1 y^2\,dx},$$

and

$$m_{\alpha,\beta,\gamma} \geqslant m.$$

Following the proof of Theorem 2.1 [2], we obtain that the minimizing sequence of $G[y]$ converges in $H_0^1(0,1)$ to some function u and

$$\inf_{y \in H_0^1(0,1)\setminus\{0\}} G[y] = G[u] = m.$$

Similarly, the function u satisfies equation (3.1) and the integral condition (3.2). Since u is non-negative on $(0,1)$, the graph of u cannot cross the axis Ox. The touching the axis Ox is also impossible due to the existence and uniqueness theorem for the solution of the Cauchy problem, as $\gamma > 1$ and $\frac{\gamma+1}{\gamma-1} > 1$. Therefore, the function u is positive on $(0,1)$.

On $(0,1)$ the function $Q_*(x) = x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}u^{\frac{2}{\gamma-1}}$ satisfies conditions (1.3) and (1.4). Since for $Q = Q_*$ and $\lambda = m$ the function u satisfies equation (1.1), satisfies conditions (1.2), since u is continuous on $[0,1]$, positive on $(0,1)$ and its derivative u' is continuous on $(0,1)$, the function u is the first eigenfunction of problem (1.1)–(1.4) with $Q = Q_*$ and the first eigenvalue $\lambda_1(Q_*) = m$.

Then

$$m_{\alpha,\beta,\gamma} = \inf_{Q\in T_{\alpha,\beta,\gamma}} \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q,y] \leqslant \inf_{y\in H_0^1(0,1)\setminus\{0\}} R[Q_*,y] = R[Q_*,u] = G[u] = m.$$

Therefore, we obtain $m_{\alpha,\beta,\gamma} = m$.

References

[1] Yu. Egorov and V. Kondratiev, *On Spectral Theory of Elliptic Operators*. Operator Theory: Advances and Applications, 89. Birkhäuser Verlag, Basel, 1996.

[2] S. Ezhak ans M. Telnova, On one upper estimate for the first eigenvalue of a Sturm–Liouville problem with Dirichlet boundary conditions and a weighted integral condition. *Mem. Differ. Equ. Math. Phys.* **73** (2018), 55–64.