Boundary Value Problems for Systems of Difference-Algebraic Equations

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We investigate the problem of finding bounded solutions [2, 3, 5]

$$z(k)\in\mathbb{R}^n,\ \ k\in\Omega:=\{0,1,2,\dots,\omega\}$$

of linear Noetherian ($n\neq\upsilon$) boundary value problem for a system of linear difference-algebraic equations [2, 5]

$$A(k)z(k+1)=B(k)z(k)+f(k),\ \ \ell z(\,\cdot\,)=\alpha,\ \ \alpha\in\mathbb{R}^{\upsilon}; \tag{1}$$

here $A(k),\ B(k)\in\mathbb{R}^{m\times n}$ are bounded matrices and $f(k)$ are real bounded column vectors,

$$\ell z(\,\cdot\,):\mathbb{R}^n\to\mathbb{R}^{\upsilon}$$

is a linear bounded vector functional defined on a space of bounded functions. We assume that the matrix $A(k)$ is, generally speaking, rectangular: $m=n$. It can be square but singular. The problem of finding bounded solutions $z(k)$ of a boundary value problem for a linear non-degenerate

$$\det B(k)\neq 0,\ \ k\in\Omega$$

system of first-order difference equations

$$z(k+1)=B(k)z(k)+f(k),\ \ \ell z(\,\cdot\,)=\alpha\in\mathbb{R}^{\upsilon}$$

was solved by A. A. Boichuk [2]. Thus, the boundary value problem (1) is a generalization of the problem solved by A. A. Boichuk. We investigate the problem of finding bounded solutions to linear Noetherian boundary value problem for a system of linear difference-algebraic equations (1) in case

$$1\leq \operatorname{rank}\,A(k)=\sigma_0,\ \ k\in\Omega.$$

As it is known [1, 10], any $(m\times n)$-matrix $A(k)$ can be represented in a definite basis in the form

$$A(k)=R_0(k)\cdot J_{\sigma_0}\cdot S_0(k),\ \ J_{\sigma_0}:=\begin{pmatrix} I_{\sigma_0} & O\\ O & O\end{pmatrix};$$

here, $R_0(k)$ and $S_0(k)$ are nonsingular matrices. The nonsingular change of the variable

$$y(k+1)=S_0(k)z(k+1)$$

reduces system (1) to the form [11]

$$A_1(k)\varphi(k+1)=B_1(k)\varphi(k)+f_1(k); \tag{2}$$

Under the condition [10], when matrices $A_1^+(k)B_1(k)$ and column vectors $A_1^+(k)f_1(k)$, are bounded and also

$$P_{A^*}(k) \neq 0, \quad P_{A_1^*}(k) \equiv 0, \tag{3}$$

we arrive at the problem of construction of solutions of the linear difference-algebraic system

$$\varphi(k+1) = A_1^+(k)B_1(k)\varphi(k) + \mathfrak{F}_1(k, \nu_1(k)), \quad \nu_1(k) \in \mathbb{R}^{\rho_1}; \tag{4}$$

here,

$$\mathfrak{F}_1(k, \nu_1(k)) := A_1^+(k)f_1(k) + P_{A_{\varrho_1}}(k)\nu_1(k),$$

$\nu_1(k) \in \mathbb{R}^{\rho_1}$ is an arbitrary bounded vector function, $A_1^+(k)$ is a pseudoinverse (by Moore–Penrose) matrix [3]. In addition, $P_{A_1^*(k)}$ is a matrix-orthoprojector [3]:

$$P_{A_1^*}(k) : \mathbb{R}^{\sigma_0} \to \mathbb{N}(A_1^*(k)),$$

$P_{A_{\rho_1}}(k)$ is an $(\rho_0 \times \rho_1)$-matrix composed of ρ_1 linearly independent columns of the $(\rho_0 \times \rho_0)$-matrix-orthoprojector:

$$P_{A_1}(k) : \mathbb{R}^{\rho_0} \to \mathbb{N}(A_1(k)).$$

By analogy with the classification of pulse boundary-value problems [3, 6, 7] we say in the (3), provided that the matrices $A_1^+(k)B_1(k)$ and column vectors $A_1^+(k)f_1(k)$ are bounded, that, for the linear difference-algebraic system (1), the first-order degeneration holds. Thus, the following lemma is proved [11].

Lemma 1. *For the first-order degeneration difference-algebraic system* (1) *having a solution of the form*

$$z(k, c_{\rho_0}) = X_1(k)\, c_{\rho_0} + K[f(j), \nu_1(j)](k), \quad c_{\rho_0} \in \mathbb{R}^{\rho_0};$$

which depends on an arbitrary continuous vector-function $\nu_1(k) \in \mathbb{R}^{\rho_1}$, where $X_1(k)$ is a fundamental matrix, $K[f(j), \nu_1(j)](k)$ is the generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1).

Denote the vector

$$\nu_1(k) := \Psi_1(k)\gamma, \quad \gamma \in \mathbb{R}^{\theta};$$

here, $\Psi_1(k) \in \mathbb{R}^{\rho_1 \times \theta}$ is an arbitrary bounded full rank matrix. Generalized Green operator of the Cauchy problem for the linear difference-algebraic system (1) of the form

$$K\big[f(j), \nu_1(j)\big](k) = K\big[f(j)\big](k) + K\big[\Psi_1(j)\big](k)\,\gamma;$$

here,

$$K\big[\Psi_1(j)\big](k) := S_0^{-1}(k-1)P_{D_{\rho_0}}\mathcal{K}\big[\Psi_1(s))\big](k),$$

and

$$\mathcal{K}\big[\Psi_1(j)\big](0) := 0, \quad \mathcal{K}\big[\Psi_1(j)\big](1) := P_{A_{\rho_1}}(0)\Psi_1(0),$$
$$\mathcal{K}\big[\Psi_1(j)\big](2) := A_1^+(1)B_1(1)\mathcal{K}\big[\Psi_1(j)\big](1) + P_{A_{\rho_1}}(1)\Psi_1(1), \ \dots,$$
$$\mathcal{K}\big[\Psi_1(j)\big](k+1) := A_1^+(k)B_1(k)\mathcal{K}\big[\Psi_1(j)\big](k) + P_{A_{\rho_1}}(k)\Psi_1(k).$$

Denote the matrix

$$\mathcal{D}_1 := \big\{Q_1; \ell K\big[\Psi_1(j)\big](\,\cdot\,)\big\} \in \mathbb{R}^{\upsilon \times (\rho_0+\theta)}.$$

Substituting the general solution of the system of linear difference-algebraic equations (1) into the boundary condition (1), we arrive at the linear algebraic equation

$$\mathcal{D}_1\,\check{c} = \alpha - \ell K\big[A^+(j)f(j)\big](\,\cdot\,), \quad \check{c} := \operatorname{col}(c_{\rho_0}, \gamma) \in \mathbb{R}^{\rho_0+\theta}. \tag{5}$$

Equation (5) is solvable iff

$$P_{\mathcal{D}_1^*}\big\{\alpha - \ell K\big[f(j)\big](\,\cdot\,)\big\} = 0. \tag{6}$$

Here, $P_{\mathcal{D}_1^*}$ is a matrix-orthoprojector:

$$P_{\mathcal{D}_1^*}:\ \mathbb{R}^{\upsilon} \to \mathbb{N}(\mathcal{D}_1^*).$$

In this case, the general solution of equation (5)

$$\check{c} = {\mathcal{D}_1}^+\big\{\alpha - \ell K\big[f(j)\big](\,\cdot\,)\big\} + P_{\mathcal{D}_1}\,\delta, \ \ \delta \in \mathbb{R}^{\rho_0+\theta}$$

determines the general solution of the boundary-value problem (1)

$$z(k,\delta) = \big\{X_1(k); K\big[\Psi_1(j)\big](k)\big\}\mathcal{D}_1^+\big\{\alpha - \ell K\big[f(j)\big](\,\cdot\,)\big\} \\ + K\big[f(j)\big](k) + \big\{X_1(k); K\big[\Psi_1(j)\big](k)\big\}P_{\mathcal{D}_1}\,\delta.$$

Here, $P_{\mathcal{D}_1}$ is a matrix-orthoprojector:

$$P_{\mathcal{D}_1}: \mathbb{R}^{\rho_0+\theta} \to \mathbb{N}(\mathcal{D}_1).$$

Thus the following theorem is valid.

Theorem 1. *The problem of finding bounded solutions of a system of linear difference-algebraic equations* (1) *in the case of first-order degeneracy, under condition* (3)*, in the case of first-order degeneracy for a fixed full rank bounded matrix* $\Psi_1(k)$*, has a solution of the form*

$$z(k,c_{\rho_0}) = X_1(k)\,c_{\rho_0} + K[f(j),\nu_1(j)](k), \ \ c_{\rho_0} \in \mathbb{R}^{\rho_0}.$$

Under condition (6) *and only under it, the general solution of the difference-algebraic boundary value problem* (1)

$$z(k,c_r) = X_r(k)c_r + G\big[f(j);\Psi_1(j);\alpha\big](k), \ \ c_r \in \mathbb{R}^r$$

is determined by the Green operator of a difference-algebraic boundary value problem (1)

$$G\big[f(j);\Psi_1(j);\alpha\big](k) := K\big[f(j)\big](k) + \big\{X_1(k); K\big[\Psi_1(j)\big](k)\big\}\mathcal{D}_1^+\big\{\alpha - \ell K\big[f(j)\big](\,\cdot\,)\big\}.$$

The matrix $X_r(k)$ *is composed of* r *linearly independent columns of the matrix*

$$\big\{X_1(k); K\big[\Psi_1(j)\big](k)\big\}P_{\mathcal{D}_1}.$$

Under condition $P_{\mathcal{D}_1^*} \neq 0$, we say that the difference-algebraic boundary-value problem (1) in the case of first-order degeneracy is a critical case, and vice versa: under condition $P_{Q_1^*} \neq 0$, $P_{\mathcal{D}_1^*} = 0$, we say that the difference-algebraic boundary-value problem (1) is reduced to the non-critical case.

The proposed scheme of studies of difference-algebraic boundary-value problems can be transferred analogously to [2–4, 9] onto nonlinear difference-algebraic boundary-value problems. On the other hand, in the case of nonsolvability, the difference-algebraic boundary-value problems can be regularized analogously [8, 12].

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