Asymptotic Representations of Solutions of Second Order Differential Equations with Nonlinearities, that are in Some Sense Near to Regularly Varying Functions

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We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')f(y,y'). \tag{1}$$

Here $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ $(-\infty < a < \omega \leq +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0,+\infty[$ are continuous functions, $f : \Delta_{Y_0} \times \Delta_{Y_1} \to]0,+\infty[$ is a continuously differentiable function, $Y_i \in \{0,\pm\infty\}$, Δ_{Y_i} is a one-sided neighborhood of Y_i, $i \in \{0,1\}$. We suppose also that every function $\varphi_i(z)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0+\sigma_1 \neq 1$ and the function f satisfies the condition

$$\lim_{\substack{v_k \to Y_k \\ v_k \in \Delta_{Y_k}}} \frac{v_k \cdot \frac{\partial f}{\partial v_k}(v_0,v_1)}{f(v_0,v_1)} = 0 \ \text{ uniformly in } \ v_j \in \Delta_{Y_j}, \ \ j \neq k, \ \ k,j \in \{0,1\}. \tag{2}$$

Many works (see, e.g., [3,4,6]) have been devoted to the establishing of asymptotic representation of solutions of equations of the form (1) in case $f \equiv 1$. In the work, the right part of (1) was or in explicit form or asymptotically represented as the product of expressions, each of which depends only of t, or only of y, or only of y'. The fact is of the most importance. In general case equation (1) can contain nonlinearities of another types, for example, $e^{|\gamma \ln|y| + \mu \ln|y'||^{\alpha}}$, $0<\alpha<1$, $\gamma,\mu \in \mathbb{R}$.

Definition. The solution y of equation (1) is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution if it is defined on $[t_0,\omega[\subset [a,\omega[$ and for all $i \in \{0,1\}$

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i, \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

The $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0-1}$ if $\lambda_0 \in R \setminus \{0,1\}$.

We need the next subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t-\omega & \text{as } \ \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \ \ i = \in \{0,1\},$$

$$J_1(t) = |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \operatorname{sign} y_1^0 \int_{B_\omega^1}^{t} |\pi_\omega(\tau)p(\tau)|^{\frac{1}{1-\sigma_1}}\, d\tau,$$

$$B_\omega^1 = \begin{cases} b & \text{if } \displaystyle\int_b^\omega |\pi_\omega(\tau)p(\tau)|^{\frac{1}{1-\sigma_1}}\, d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_b^\omega |\pi_\omega(\tau)p(\tau)|^{\frac{1}{1-\sigma_1}}\, d\tau < +\infty, \end{cases}$$

$$I_1(t)=\alpha_0\Big|\frac{\lambda_0-1}{\lambda_0}\Big|^{\sigma_0}\int\limits_{A^1_\omega}^{t}\frac{(\tau)}{|\pi_\omega(\tau)|^{-\sigma_0}}\,d\tau,\quad A^1_\omega=\begin{cases} a & \text{if } \displaystyle\int\limits_a^\omega \frac{p(\tau)}{|\pi_\omega(\tau)|^{-\sigma_0}}\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_a^\omega \frac{p(\tau)}{|\pi_\omega(\tau)|^{-\sigma_0}}\,d\tau<+\infty,\end{cases}$$

$$J_2(t)=|\sigma_0|^{-\frac{1}{\sigma_0}}\operatorname{sign}y^0_1\int\limits_{B^2_\omega}^{t}|I_1(\tau)|^{-\frac{1}{\sigma_0}}\,d\tau,\quad B^2_\omega=\begin{cases} b & \text{if } \displaystyle\int\limits_b^\omega |I_1(\tau)|^{-\frac{1}{\sigma_0}}\,d\tau=+\infty,\\ \omega & \text{if } \displaystyle\int\limits_b^\omega |I_1(\tau)|^{-\frac{1}{\sigma_0}}\,d\tau<+\infty.\end{cases}$$

Theorem 1. *Let $\sigma_1\neq 1$. Then for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation* (1)*, where $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, the next conditions are necessary*

$$\pi_\omega(t)y^0_1y^0_0\lambda_0(\lambda_0-1)>0,\quad \pi_\omega(t)\alpha_0y^0_1(\lambda_0-1)>0\ \text{ as }\ t\in[a,\omega[\,, \tag{3}$$

$$\lim_{t\uparrow\omega}y^0_0|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}=Y_0,\quad \lim_{t\uparrow\omega}y^0_1|\pi_\omega(t)|^{\frac{1}{\lambda_0-1}}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_1'(t)}{J_1(t)}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\cdot\frac{\lambda_0}{\lambda_0-1}\,. \tag{4}$$

If

$$\lambda_0\neq\sigma_1-1\ \text{ or }\ (\sigma_1-1)(\sigma_0+\sigma_1-1)>0, \tag{5}$$

conditions (3), (4) *are sufficient for the existence of such solutions of equation* (1).

For $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) *the next asymptotic representations take place as $t\uparrow\omega$,*

$$\begin{gathered}\frac{y(t)|y(t)|^{-\frac{\sigma_0}{1-\sigma_1}}}{(f(y(t),y'(t))\Theta_0(y(t))\Theta_1(y'(t)))^{\frac{1}{1-\sigma_1}}}=\frac{1-\sigma_0-\sigma_1}{1-\sigma_1}\,J_1(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)].\end{gathered} \tag{6}$$

By conditions (3), (5) and the first of the asymptotic representations (6), obtained in Theorem 1, it is clear that the case $\sigma_1=1$ requires a separate investigation. The following theorem covers this case.

Theorem 2. *Let $\sigma_1=1$. Then for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation* (1)*, where $\lambda_0\in\mathbb{R}\setminus\{0,1\}$, the next conditions are necessary and sufficient*

$$y^0_0J_2(t)>0,\ \ \alpha_0y^0_0\lambda_0>0,\ \ y^0_1\sigma_0I_1(t)<0\ \text{ as }\ t\in[a,\omega[\,,$$

$$\lim_{t\uparrow\omega}y^0_1|I_1(t)|^{-\frac{1}{\sigma_0}}=Y_1,\quad \lim_{t\uparrow\omega}y^0_0|\pi_\omega(t)|^{\frac{\lambda_0}{\lambda_0-1}}=Y_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_1'(t)}{I_1(t)}=\frac{\sigma_0}{1-\lambda_0}\,.$$

For $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) *the next asymptotic representations take place as $t\uparrow\omega$,*

$$\begin{gathered}y(t)|\Theta_0(y(t))\Theta_1(y'(t))f(y(t),y'(t))|^{\frac{1}{\sigma_0}}=J_2(t)[1+o(1)],\\ \frac{y'(t)}{y(t)}=\frac{\lambda_0}{(\lambda_0-1)\pi_\omega(t)}\,[1+o(1)].\end{gathered}$$

For the equations of the form (1) the existence of different types of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions essentially depends from the orders σ_0 and σ_1 of the regularly varying functions φ_0, φ_1 as their arguments tend to Y_0, Y_1 respectively, and from the type of function p, that as must be mentioned, does not necessary have to be a regularly varying. By the results of Theorem 1, precisely by the third condition of (3), it is clear that $P_\omega(Y_0, Y_1, \lambda_0)$-solutions for which $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ may appear in the equations of the form (1), when p is regularly varying function as $t \uparrow \omega$. To simplify the calculations, we take $p(t) \equiv t^\gamma$. On the interval $[t_0; +\infty[$ $(t_0 > 0)$ we consider the differential equation

$$y'' = t^\gamma \varphi_0(y)\varphi_1(y') \exp\big(|\ln|y||^{\mu_0} + |\ln|y'||^{\mu_1}\big)^{\mu_2}, \tag{7}$$

where $\gamma \in \mathbb{R} \setminus \{0\}$, $\mu_i \in (0,1)$ for each $i \in \{0,1,2\}$. This equation is of the form (1), with $\alpha_0 = 1$, $p(t) = t^\gamma$, $f(y, y') = \exp(|\ln|y||^{\mu_0} + |\ln|y'||^{\mu_1})^{\mu_2}$. Now

$$\Delta_{Y_k} = [y_k^0, +\infty[\ \ (\forall\, k \in \{0,1\}), \quad \omega = Y_0 = Y_1 = +\infty,$$

$$J_1(t) = \frac{1-\sigma_1}{\gamma - \sigma_1 + 2}\, |\lambda_0 - 1|^{\frac{1}{1-\sigma_1}} \operatorname{sign} y_1^0 t^{\frac{\gamma-\sigma_1+2}{1-\sigma_1}}, \quad I_1(t) = \Big|\frac{\lambda_0 - 1}{\lambda_0}\Big|^{\sigma_0} \frac{t^{\sigma_0+\gamma+1}}{\sigma_0 + \gamma + 1},$$

$$J_2(t) = -\Big(\frac{\sigma_0 + \gamma + 1}{|\sigma_0|}\Big)^{\frac{1}{\sigma_0}} \frac{\sigma_0}{\gamma+1} \operatorname{sign} y_1^0 t^{-\frac{\gamma+1}{\sigma_0}}.$$

Condition (2) in our case takes the following form

$$\lim_{\substack{v_k \to Y_k \\ v_k \in \Delta_{Y_k}}} \mu_k \mu_2 |\ln|v_k||^{\mu_k - 1}\big(|\ln|v_0||^{\mu_0} + |\ln|v_1||^{\mu_1}\big)^{\mu_2 - 1} = 0,$$

where $k \in \{0,1\}$.

It is clear that since $m_i - 1 < 0$ for all $i \in \{0,1,2\}$, the function under the sign of a limit tends to zero uniformly over $v_j \in [y_k^0; +\infty[$, $j \neq k$, $k, j \in \{0,1\}$.

We apply Theorem 1 and obtain that from all $P_{+\infty}(+\infty, +\infty, \lambda_0)$-solutions, where $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$, equation (7) can have only $P_{+\infty}(+\infty, +\infty, \frac{\gamma-\sigma_1+2}{\gamma+\sigma_1+1})$-solutions if

$$(\gamma - \sigma_1 + 2)(1 - \sigma_0 - \sigma_1) > 0, \quad \frac{1 - \sigma_0 - \sigma_1}{\gamma + \sigma_0 + 1} > 0.$$

These conditions are necessary, and if, together to them,

$$\frac{\gamma - \sigma_1 + 2}{\gamma + \sigma_0 + 1} \neq \sigma_1 + 1 \ \text{ or } \ (\sigma_1 - 1)(\sigma_0 + \sigma_1 - 1) > 0,$$

they are sufficient for the existence of such solutions of equation (7). In addition, for each such $P_{+\infty}(+\infty, +\infty, \frac{\gamma-\sigma_1+2}{\gamma+\sigma_1+1})$-solution of equation (7) the following asymptotic representations take place as $t \to +\infty$,

$$\frac{(y(t))^{1-\sigma_1}|y'(t)|^{\sigma_1}}{\varphi_0(y(t))\varphi_1(y'(t)) \exp(|\ln|y(t)||^{\mu_0} + |\ln|y'(t)||^{\mu_1})^{\mu_2}} = \Big(\frac{1-\sigma_0-\sigma_1}{\gamma-\sigma_1+2}\Big)^{1-\sigma_1} \Big|\frac{1-\sigma_0-\sigma_1}{\gamma+\sigma_0+1}\Big| t^{\gamma-\sigma_1+2}[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = \frac{\gamma - \sigma_1 + 2}{1 - \sigma_0 - \sigma_1} \cdot \frac{1}{t}\, [1 + o(1)].$$

Then we also consider the differential equation (7) under the assumption that $\sigma_1 = 1$. We apply Theorem 2 and find that in this case from the $P_{+\infty}(+\infty, +\infty, \lambda_0)$-solutions, where $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$, equation (7) can have only $P_{+\infty}(+\infty, +\infty, \frac{\gamma+1}{\sigma_0+\gamma+1})$-solutions if

$$y_1^0 \sigma_0 < 0, \ \ y_0^0 \frac{\gamma + 1}{\sigma_0 + \gamma + 1} > 0.$$

This condition is necessary and sufficient for the existence of such solutions of equation (7). In addition, for any $P_{+\infty}(+\infty, +\infty, \frac{\gamma+1}{\sigma_0+\gamma+1})$-solution of equation (7) the following asymptotic representations take place as $t \to +\infty$,

$$y(t)\exp\Big(\frac{(|\ln|y||^{\mu_0}+|\ln|y'||^{\mu_1})^{\mu_2}}{\sigma_0}\Big) = -\Big(\frac{\sigma_0+\gamma+1}{|\sigma_0|}\Big)^{\frac{1}{\sigma_0}}\,\frac{\sigma_0}{\gamma+1}\,\operatorname{sign} y_1^0 t^{-\frac{\gamma+1}{\sigma_0}}\,[1+o(1)],$$

$$\frac{y'(t)}{y(t)} = -\frac{\gamma+1}{\sigma_0 t}\,[1+o(1)].$$

Another classes of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) have also been investigated before (see, e.g., [5]). The sufficiently important class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equations like (1) has been considered only for cases when $f(y, y') \equiv 1$ and the function $\varphi_0(z)|z|^{-\sigma_0}$ satisfies some additional conditions. Later it has appeared an opportunity to extend the results onto more general cases (see, e.g., [1]). But functions that contain in their left side the derivative of the unknown function as it is in general case of equation (1), haven't been considered before. Let us notice that the derivative of every $P_\omega(Y_0, Y_1, \pm\infty)$-solution is a slowly varying function as $t \uparrow \omega$. It makes a lot of difficulties for the investigations. The sufficiently important class of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1) is established (see, [2]) for the case $f(y, y') \equiv \exp(R(|\ln|yy'||))$, $R :]0; +\infty[\, \to\,]0; +\infty[$ is continuously differentiable function, that is regularly varying on infinity of the order μ, $0 < \mu < 1$ and has monotone derivative.

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