Asymptotic Behaviour of \( P_\omega(Y_0,0) \)-Solutions of Second-Order Nonlinear Differential Equations with Regularly and Rapidly Varying Nonlinearities

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Consider the differential equation

\[
y'' = \sum_{i=1}^{m} \alpha_i p_i(t) \varphi_i(y),
\]

where \( \alpha_i \in \{-1,1\} \) (\( i = \overline{1,m}, p_i : [a,\omega[ \to ]0, +\infty[ \) (\( i = \overline{1,m} \)) are continuous functions, \( \omega \leq \infty \), \( \varphi_i : \Delta_{Y_0} \to ]0, +\infty[ \) (\( i = \overline{1,m} \)), where \( \Delta_{Y_0} \) is a one-sided neighborhood of \( Y_0 \), \( Y_0 \) is equal either to zero or \( \pm \infty \), are continuous functions for \( i = \overline{1,l} \) and twice continuously differentiable for \( i = \overline{l+1,m} \), and for each \( i \in \{1,\ldots,l\} \) for some \( \sigma_i \in \mathbb{R} \)

\[
\lim_{y \to Y_0 \atop y \in \Delta_{Y_0}} \frac{\varphi_i(\lambda y)}{\varphi_i(y)} = \lambda^{\sigma_i} \text{ for any } \lambda > 0,
\]

and for each \( i \in \{l+1,\ldots,m\} \)

\[
\varphi_i'(y) \neq 0 \text{ as } y \in \Delta_{Y_0}, \quad \lim_{y \to Y_0 \atop y \in \Delta_{Y_0}} \varphi_i(y) \in ]0, +\infty[, \quad \lim_{y \to Y_0 \atop y \in \Delta_{Y_0}} \frac{\varphi_i''(y)}{\varphi_i'(y)} = 1.
\]

It follows from the conditions (2) and (3) that \( \varphi_i \) (\( i = \overline{1,l} \)) are regularly varying functions, as \( y \to Y_0 \), of orders \( \sigma_i \) and \( \varphi_i \) (\( i = \overline{l+1,m} \)) are rapidly varying functions, as \( y \to Y_0 \) (see [4, Introduction, pp. 2, 4]).

**Definition.** A solution \( y \) of the differential equation (1) is called \( P_\omega(Y_0,\lambda_0) \) – solution, where \( -\infty \leq \lambda_0 \leq +\infty \), if it is defined on some interval \( [t_0,\omega[ \subset [a,\omega[ \) and satisfies the following conditions

\[
\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y'(t) = \begin{cases} 0, & \text{either} \\ \pm \infty, & \text{or} \end{cases}, \quad \lim_{t \uparrow \omega} \frac{y''(t)}{y'(t)y(t)} = \lambda_0.
\]

By its asymptotic properties, the class of \( P_\omega(Y_0,\lambda_0) \) – solutions is split into 4 non-intersecting subsets that correspond to the next value of the parameter \( \lambda_0 \)

\[
\lambda_0 \in \mathbb{R} \setminus \{0,1\}, \quad \lambda_0 = 1, \quad \lambda_0 = 0, \quad \lambda_0 = \pm \infty.
\]

The existence conditions of \( P_\omega(Y_0,\lambda_0) \) – solutions of the differential equation (1) and asymptotic representations, as \( t \uparrow \omega \), of such solutions and their first-order derivatives, are established for each of these cases in the case where, for some \( s \in \{1,\ldots,m\} \)

\[
\lim_{t \uparrow \omega} \frac{p_i(t)\varphi_i(y(t))}{p_s(t)\varphi_s(y(t))} = 0 \text{ for all } i \in \{1,\ldots,m\} \setminus \{s\}, \quad (4)
\]
i.e., where the right-hand side of Eq. (1) for each such solution \( y \) is equivalent for \( t \uparrow \omega \) to one term with regularly or rapidly varying nonlinearity (see [1–3]).

In this paper, we formulate the main results obtained for the case \( \lambda_0 = 0 \).

Let

\[
\Delta y_0 = \Delta y_0(b), \quad \text{where} \quad \Delta y_0(b) = \begin{cases} [b, Y_0[ & \text{if } \Delta y_0 \text{ is a left neighborhood of } Y_0, \\ ]Y_0, b] & \text{if } \Delta y_0 \text{ is a right neighborhood of } Y_0, \end{cases}
\]

and the number \( b \) satisfy the inequalities

\[ |b| < 1 \quad \text{as } Y_0 = 0 \quad \text{and} \quad b > 1 \quad (b < -1) \quad \text{as } Y_0 = +\infty \quad (Y_0 = -\infty). \]

We set

\[
\nu_0 = \text{sign } b, \quad \nu_1 = \begin{cases} 1, & \text{if } \Delta y_0(b) = [b, Y_0[, \\
-1, & \text{if } \Delta y_0(b) = ]Y_0, b]. \end{cases}
\]

\[
\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\
-t, & \text{if } \omega < +\infty, \end{cases}
\]

\[
J_{1s}(t) = \int_{A_{1s}} p_s(\tau) \, d\tau, \quad J_{2s}(t) = \int_{A_{2s}} J_{1s}(\tau) \, d\tau, \quad J_{3s}(t) = \int_{A_{3s}} \pi_\omega(\tau)p_0(\tau) \, d\tau,
\]

\[
H_s(y) = \int_{B_s} \frac{du}{\varphi_s(u)} , \quad B_s = \begin{cases} b, & \text{if } \int b \, dy = \pm \infty, \\
Y_0, & \text{if } \int b \, dy = \text{const}. \end{cases}
\]

\[
Z_s = \lim_{y \to Y_0} H_s(y),
\]

\[
J_{\varphi}(t) = \int_{A_{\varphi}} p_0(\tau)\varphi_s(H_s^{-1}(\alpha_s J_{3s}(\tau))) \, d\tau, \quad E_s(t) = \alpha_s \pi_\omega^2(t)p_0(\tau)\varphi_s'(H_s^{-1}(\alpha_s J_{3s}(t))),
\]

\[
G_s(t) = \frac{y\varphi_s(y)}{\varphi_s(y)} \bigg|_{y = H_s^{-1}(\alpha_s J_{3s}(t))}, \quad \Phi_s(t) = \frac{y\varphi_s'(y)}{\varphi_s(y)} \bigg|_{y = H_s^{-1}(\alpha_s J_{3s}(t))},
\]

\[
\mu_s = \text{sign } \varphi_s'(y), \quad \gamma_s = \lim_{t \uparrow \omega} \frac{E_s(t)\Phi_s(t)}{G_s(t)} , \quad \psi_s(t) = \int_{t_0}^t \frac{|E_s(\tau)|^2}{\pi_\omega(\tau)} \, d\tau,
\]

where \( s \in \{1, \ldots, m\} \), \( p_0 : [a, \omega[ \to ]0, +\infty[ \) are continuous functions so that \( p_0(t) \sim p_0(t) \) as \( t \uparrow \omega \), every limit of integration \( A_{1s}, A_{2s}, A_{3s}, A_{\varphi} \) is equal to either \( a \) or \( \omega \) and is chosen so that the corresponding integral tends either to \( \pm \infty \), or to zero with \( t \uparrow \omega \), \( t_0 \) is some number of \( [a, \omega[ \).

**Theorem 1.** Let \( \sigma_s \neq 1 \) for some \( s \in \{1, \ldots, l\} \) and there exist finite or equal to infinity limit

\[
\lim_{t \uparrow \omega} \pi_\omega(t)J_{1s}(t).
\]

For existence of \( P_\omega(Y_0, 0) \) – solutions of equation (1), satisfied the limit relations (4), it is necessary that the inequalities

\[
\alpha_s\nu_0(1 - \sigma_s)J_{2s}(t) > 0, \quad \alpha_s\nu_1\pi_\omega(t) < 0 \quad \text{as } t \in ]a, \omega[.
\]
and conditions
\[
\alpha_s \lim_{t \uparrow \omega} J_{2s}(t) = Z_s, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J_{1s}(t)}{J_{2s}(t)} = -1, \quad \lim_{t \uparrow \omega} \frac{J_{2s}^2(t)}{p_s(t) J_{2s}(t)} = 0, \tag{6}
\]
\[
\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)))}{p_s(t) \varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \quad \text{for all } i \in \{1, \ldots, l\} \setminus \{s\}, \tag{7}
\]
\[
\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(H_s^{-1}(\alpha_s J_{2s}(t))(1 + \delta_i))}{p_s(t) \varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \quad \text{for all } i \in \{l + 1, \ldots, m\}
\]
hold, where \(\delta_i\) are arbitrary numbers of a one-sided neighborhood of zero. Moreover, for each of such solutions the following asymptotic representations hold
\[
y(t) = H_s^{-1}(\alpha_s J_{2s}(t))[1 + o(1)] \quad \text{at} \quad t \uparrow \omega, \tag{8}
\]
\[
y'(t) = \frac{J_{1s}(t) H_s^{-1}(\alpha_s J_{2s}(t))}{(1 - \sigma_s) J_{2s}(t)} [1 + o(1)] \quad \text{at} \quad t \uparrow \omega. \tag{9}
\]

**Theorem 2.** Let \(\sigma_s \neq 1\) for some \(s \in \{1, \ldots, l\}\), conditions (5)–(7) hold and
\[
\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(H_s^{-1}(\alpha_s J_{2s}(t)(1 + u)))}{p_s(t) \varphi_s(H_s^{-1}(\alpha_s J_{2s}(t)))} = 0 \quad \text{for all } i \in \{l + 1, \ldots, m\}
\]
uniformly with respect to \(u \in [-\delta, \delta]\) for any \(0 < \delta < 1\). Then the differential equation (1) has \(P_\omega(Y_0, 0)\) – solutions that admit the asymptotic representations (8) and (9). Moreover, if \(\alpha_s \nu_0(1 - \sigma_s) \pi_\omega(t) < 0\) as \(t \in ]a, \omega[, \) there is a one-parameter family of such solutions in case \(\omega = +\infty\) and two-parameter family in case \(\omega < +\infty\).

**Theorem 3.** Let for some \(s \in \{l + 1, \ldots, m\}\) the function \(p_s\) might be representable in the form
\[
p_s(t) = p_{0s}(t)[1 + r_s(t)], \quad \text{where} \quad \lim_{t \uparrow \omega} r_s(t) = 0, \tag{10}
\]
\(p_{0s} : [a, \omega[ \rightarrow ]0, +\infty[\) is a continuously differentiable function, \(r_s : [a, \omega[ \rightarrow ]-1, +\infty[\) is a continuous function, and let the conditions
\[
\frac{\varphi_s(y) \varphi'_i(y)}{\varphi'_s(y) \varphi_i(y)} = O(1) \quad (i = l + 1, m) \quad \text{for} \quad y \rightarrow Y_0 \tag{11}
\]
hold. Then, for the existence of \(P_\omega(Y_0, 0)\) – solutions of the differential equation (1) satisfying conditions (4), it is necessary that, there exist finite or equal to infinity limit
\[
\lim_{t \uparrow \omega} \frac{\pi_\omega(t) \varphi'_s(t)}{J_{\varphi_s}(t)},
\]
and the conditions
\[
\alpha_s \nu_1 \pi_\omega(t) < 0, \quad \alpha_s \mu_s J_{3s}(t) > 0 \quad \text{as} \quad t \in ]a, \omega[, \tag{12}
\]
\[
-\alpha_s \lim_{t \uparrow \omega} J_{3s}(t) = Z_s, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'_{\varphi_s}(t)}{J_{\varphi_s}(t)} = -1, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega^2(t) p_{0s}(t) \varphi_s(H_s^{-1}(\alpha_s J_{3s}(t)))}{H_s^{-1}(\alpha_s J_{3s}(t))} = 0, \tag{13}
\]
\[
\lim_{t \uparrow \omega} \frac{p_i(t) \varphi_i(H_s^{-1}(\alpha_s J_{3s}(t))))}{p_s(t) \varphi_s(H_s^{-1}(\alpha_s J_{3s}(t)))} = 0 \quad \text{for all } i \in \{1, \ldots, m\} \setminus \{s\}
\]
be satisfied. Moreover, each such solutions has the asymptotic representations
\[
y(t) = H_s^{-1}(\alpha_s J_{3s}(t))[1 + o(1) G_s(t)] \quad \text{at} \quad t \uparrow \omega, \tag{15}
\]
\[
y'(t) = -\alpha_s \pi_\omega(t) p_{0s}(t) \varphi_s(H_s^{-1}(\alpha_s J_{3s}(t)))[1 + o(1)] \quad \text{at} \quad t \uparrow \omega. \tag{16}
\]
Theorem 4. Let for some $s \in \{l + 1, \ldots, m\}$ the conditions (10), (11), (12)–(14) be satisfied and
\[
\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J'_s(t)}{J_{s_3}(t)} = \eta_s, \quad \text{where } \eta_s \in \mathbb{R}.
\]
Then:

1) if $\eta_s > 0$ or $\eta_s = 0$ and $\alpha_s \mu_s = 1$, the differential equation (1) has a one-parameter family of $P_\omega(Y_0, 0)$ – solutions with the asymptotic representations (15) and (16);

2) if $\eta_s < 0$ or $\eta_s = 0$ and $\alpha_s \mu_s = -1$, there is a two-parameter family of $P_\omega(Y_0, 0)$ – solutions which admit the asymptotic representations (15), (16) in case $\omega = +\infty$ and there is at least one such solution in case $\omega = +\infty$.

Theorem 5. Let for some $s \in \{l + 1, \ldots, m\}$ the function $p_s$ be representable in the form (10), let conditions (11), (12)–(14) hold, and let the limits (which are finite or equal to $\pm \infty$)
\[
\lim_{t \uparrow \omega} \frac{\pi_\omega(t) J''_{s}(t)}{J''_{s}(t)}, \quad \lim_{y \to Y_0} \frac{(\varphi'(y))' (\varphi'(y))^2}{\varphi'(y)^2}, \quad \lim_{t \uparrow \omega} \frac{E_s(t) \Phi_s(t)}{G_s(t)}, \quad \lim_{t \uparrow \omega} \psi''_s(t) \psi_2(t)
\]
exist. Then:

1) if $\alpha_s \mu_s = 1$, the differential equation (1) has a one-parameter family of $P_\omega(Y_0, 0)$ – solutions which admit the asymptotic representations (15) and (16) and are such that their derivatives satisfy the asymptotic relation
\[
y'(t) = -\alpha_s \pi_\omega(t) p_{0s}(t) \varphi_s(H_s^{-1}(-\alpha_s J_{s_3}(t))) \left[1 + |E_s(t)|^{-\frac{1}{2}} o(1)\right] \quad \text{at } t \uparrow \omega;
\]

2) if $\alpha_s \mu_s = -1$ and
\[
\gamma_s \neq \frac{1}{3}; \quad \lim_{t \uparrow \omega} \psi_s(t) r_s(t) = 0, \quad \lim_{t \uparrow \omega} \psi_s^2(t) \left[r_s(t) + 2 + \frac{\pi_\omega(t) J''_{s}(t)}{J''_{s}(t)}\right] = 0,
\]
\[
\lim_{t \uparrow \omega} \frac{\psi_s(t)}{E_s(t)} = 0 \quad \text{at } \gamma_s = 0, \quad \lim_{t \uparrow \omega} \psi_s^2(t) \sum_{i=1}^{m} p_i(t) \varphi_i(H_s^{-1}(-\alpha_s J_{s_3}(t))) = 0,
\]

the differential equation (1) has a $P_\omega(Y_0, 0)$ – solution with asymptotic representations
\[
y(t) = H_s^{-1}(-\alpha_s J_{s_3}(t)) \left[1 + \frac{o(1)}{G_s(t) \psi_s(t)}\right] \quad \text{at } t \uparrow \omega,
\]
\[
y'(t) = -\alpha_s \pi_\omega(t) p_{0s}(t) \varphi_s(H_s^{-1}(-\alpha_s J_{s_3}(t))) \left[1 + |E_s(t)|^{-\frac{1}{2}} \psi_s^{-1}(t) o(1)\right] \quad \text{at } t \uparrow \omega.
\]

Moreover, there exists a two-parameter family of such solutions in case when $\gamma_s \in (0, 1/3)$ or $\gamma_s = 0$ and $\alpha_s \nu_1 = 1$.

References

