Kneser Solutions to Second Order Nonlinear Equations with Indefinite Weight

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1 Introduction

Consider the nonlinear differential equation

\[(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad t \in [1, \infty),\]

(1.1)

where

\[\Phi(u) := |u|^{\alpha} \text{sgn } u, \quad \alpha > 0.\]

We study the problem of the existence of Kneser solutions, that is solutions \(x\) such that

\[x(t) > 0, \quad x'(t) < 0 \quad \text{for } t \in [1, \infty),\]

(1.2)

satisfying the boundary conditions

\[x(1) = c > 0, \quad \lim_{t \to \infty} x(t) = 0.\]

(1.3)

We assume that the functions \(a, b\) are continuous functions on \([1, \infty), a(t) > 0\), and

\[J_a = \int_{1}^{\infty} \Psi\left(\frac{1}{a(t)}\right) dt < \infty,
\]

where \(\Psi\) is the inverse function of \(\Phi\), that is \(\Psi(u) := |u|^{1/\alpha} \text{sgn } u\). The weight function \(b\) is bounded from above and is allowed to change sign (in)finite many times. The nonlinearity \(F\) is a continuous function on \([0, \infty)\) such that \(F(u) > 0\) for \(u > 0\) and

\[\limsup_{u \to 0^+} \frac{F(u)}{\Phi(u)} < \infty.\]

(1.4)

This problem is motivated by [3] where some asymptotic BVPs are studied for (1.1) in case \(F(u) = |u|^{\beta} \text{sgn } u, \beta > 0\) and \(b(t) \leq 0\) for \(t \geq 1\). There are few contributions to the solvability of the boundary value problems when the function \(b\) is allowed to change its sign. For example, the boundary value problem on the compact interval with the indefinite weight has been considered in [1].

In [4], our method used here is based on a fixed point theorem for operators defined in a Fréchet space stated in [2]. This approach does not require the explicit form of the fixed point operator but only good \(a\text{-priori}\) bounds. These bounds are obtained using the principal solutions of an associated linear or half-linear differential equations.

Our proofs are based on the following fixed point theorem.
Theorem 1 ([2]). Consider the BVP on \([1, \infty)\),
\[(a(t)\Phi(x'))' + b(t)F(x) = 0, \quad x \in S,\]  
(1.5)
where \(S\) is a nonempty subset of the Fréchet space \(C[1, \infty)\) of the continuous functions defined in \([1, \infty)\) endowed with the topology of uniform convergence on compact subsets of \([1, \infty)\).

Let \(G\) be a continuous function on \([0, \infty) \times [0, \infty)\) such that \(F(d) = G(d, d)\) for any \(d \in [0, \infty)\). Assume that there exist a nonempty, closed, convex and bounded subset \(\Omega \subseteq C[1, \infty)\) and a bounded closed subset \(S_1 \subseteq S \cap \Omega\) such that for any \(u \in \Omega\) the BVP on \([1, \infty)\)
\[(a(t)\Phi(x'))' + b(t)G(u(t), x(t)) = 0, \quad x \in S_1\]
admits a unique solution. Then the BVP (1.5) has at least a solution.

In the sequel, we introduce the notion of principal solution and disconjugacy for the half-linear equation
\[(a(t)\Phi(y'))' + \beta(t)\Phi(y) = 0,\]  
(1.6)
where \(\beta\) is a continuous function for \(t \geq 1\). When (1.6) is nonoscillatory, the notion of principal solution of (1.6) has been introduced in [7] by following the Riccati approach, see, also [6, Sections 2.2, 4.2]. Among all eventually different from zero solutions of the associated Riccati equation
\[u' + \beta(t) + R(t, u) = 0,\]  
(1.7)
where
\[R(t, u) = \alpha |w| \Psi \left( \frac{|w|}{a(t)} \right),\]
there exists one, say \(w_x\), which is continuable to infinity and is minimal in the sense that any other solution \(w\) of (1.7), which is continuable to infinity, satisfies \(w_x(t) < w(t)\) as \(t \to \infty\). This concept extends to the half-linear case the well-known notion of principal solution that was introduced in 1936 by W. Leighton and M. Morse for the linear case.

We recall that (1.6) is said to be disconjugate on an interval \(I \subset [T, \infty)\) if any nontrivial solution of (1.6) has at most one zero on \(I\). Equation (1.6) is disconjugate on \([T, \infty)\) if and only if it has the principal solution without zeros on \((T, \infty)\).

An important role in our considerations is played by a comparison theorem for the principal solutions of Sturm majorant and minorant half-linear equations established in [5]. It is worth to note that if \(\alpha = 1\), the half-linear equation reduces to linear one and its principal solution can be characterized by the condition
\[\int_1^\infty \frac{1}{a(t)x^2(t)} dt = \infty.\]  
(1.8)
However, the integral characterization of the principal solution of half-linear equations remains an open problem. Hence, in the half-linear case a different approach has been used.

2 Existence and uniqueness theorem: case \(\alpha = 1\)

Consider nonlinear equation with the Sturm–Liouville operator
\[(a(t)x')' + b(t)F(x) = 0.\]  
(2.1)
In addition to assumptions stated in Introduction, we also assume here that $F$ is differentiable on $[0, \infty)$ with bounded nonnegative derivative, that is
\begin{equation}
0 \leq \frac{dF(u)}{du} \leq K \text{ for } u \geq 0,
\end{equation}
and satisfies
\begin{equation}
\lim_{u \to 0^+} \frac{F(u)}{u} = k_0, \quad \lim_{u \to \infty} \frac{F(u)}{u} = k_{\infty},
\end{equation}
where $0 \leq k_0 \neq k_{\infty}$.

The following result has been stated in [4, Theorem 3], see also Remark 5.

**Theorem 2.** Let $B > 0$ be such that
\begin{equation}
b(t) \leq B \text{ on } [1, \infty)
\end{equation}
and assume that the linear differential equation
\begin{equation}
v'' + \frac{BK}{a(t)} v = 0
\end{equation}
is disconjugate on $[1, \infty)$. Then, for any $c > 0$, equation (2.1) has a unique solution $x$ satisfying (1.2) and (1.3). Moreover, such solution $x$ satisfies (1.8).

**Example.** Consider the equation
\begin{equation}
(t^2 x')' + \frac{1}{4} \cos \left( \frac{\pi t}{2} \right) F(x) = 0 \quad (t \geq 1),
\end{equation}
where
\begin{equation}
F(u) = \frac{u}{1 + \sqrt{u}}.
\end{equation}
Then $F$ satisfies (2.2), (2.3), $K = 1$ and $b(t) \leq 1/4$ for $t \geq 1$. Hence equation (2.4) becomes the Euler equation
\begin{equation}
v'' + \frac{1}{4t^2} v = 0 \quad (t \geq 1),
\end{equation}
which has a principal solution $v = \sqrt{t}$ and thus it is disconjugate on $[1, \infty)$. By Theorem 2, for any $c > 0$, equation (2.5) has a unique Kneser solution satisfying (1.2), (1.3) and (1.8).

### 3 Existence theorem in the general case

Denote by $b_+$, $b_-$, respectively, the positive and the negative part of $b$, i.e., $b_+(t) = \max\{b(t), 0\}$, $b_-(t) = -\min\{b(t), 0\}$. Thus $b(t) = b_+(t) - b_-(t)$.

Denote by $\widetilde{F}$ the function
\begin{equation}
\widetilde{F}(v) = \frac{F(v)}{\Phi(v)} \text{ on } (0, \infty).
\end{equation}
In view of (1.4), the function $\widetilde{F}$ is bounded in the neighbourhood of zero.

Using Theorem 1 and asymptotic properties of the half-linear equations, we obtain from [5, Theorem 1] the following result.
Theorem 3. Let $c > 0$ be fixed and $M_c$ be such that
\[ \bar{F}(v) \leq M_c \] on $[0, c]$.

Assume that the half-linear differential equation
\[ (a_1(t)\Phi(y'))' + \beta_1(t)\Phi(y) = 0, \] (3.2)
where
\[ a_1(t) \leq a(t), \quad \beta_1(t) \geq M_c b_+(t) \] on $t \geq 1$, (3.3)
has a principal solution which is positive decreasing on $[1, \infty)$.

Then, the BVP (1.1), (1.3) has at least one solution $x$ if any of the following conditions holds:

(i$_1$) \[ \lim_{T \to \infty} \int_1^T \frac{1}{\Psi(a(s))} ds \int_1^T b(t) dt < \infty; \] (3.4)

(i$_2$) There exists $\bar{t} \geq 1$ such that $b_+(t) = 0$ for any $t \geq \bar{t}$.

Moreover, if (i$_1$) holds, such solution $x$ satisfies
\[ \lim_{t \to \infty} \frac{x(t)}{\Psi(a^{-1}(s)) ds} = \ell, \quad 0 < \ell < \infty. \] (3.5)

Remark. A typical nonlinearity satisfying (1.4) is $F(u) = u^\alpha$. A prototype of an half-linear equation (3.2) is the Euler type equation
\[ (t^{1+\alpha} \Phi(y'))' + \left( \frac{1}{1+\alpha} \right)^{1+\alpha} \Phi(y) = 0. \] (3.6)

From [6, Theorem 4.2.4], the function
\[ y_0(t) = \left( \frac{1}{1+\alpha} \right)^{1/\alpha} t^{-1/(1+\alpha)} \]
is the principal solution of (3.6). Moreover, $y_0$ is positive decreasing on the interval $[1, \infty)$ and so (3.6) is disconjugate on the same interval. Other examples can be found in [5].

References

