On the Well-Posedness Question of the Modified Cauchy Problem for Linear Systems of Impulsive Equations with Singularities

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Let $I \subset \mathbb{R}$ be an interval non-degenerate in the point, $b_0 = \sup I$ and

$I_0 = I \setminus \{ b_0 \}$.

Consider the linear system of impulsive equations with fixed points of impulses actions

$$
\frac{dx}{dt} = P(t)x + q(t) \text{ for a.a. } t \in I_0 \setminus \{ \tau_l \}_{l=1}^{+\infty},
$$

(1)

$$
x(\tau_l+) - x(\tau_l-) = G_l x(\tau_l) + g_l \quad (l = 1, 2, \ldots),
$$

(2)

where $P \in L_{loc}(I_0, \mathbb{R}^{n \times n})$, $q \in L_{loc}(I_0, \mathbb{R}^n)$, $G_l \in \mathbb{R}^{n \times n}$ ($l = 1, 2, \ldots$), $g_l \in \mathbb{R}^n$ ($l = 1, 2, \ldots$), $\tau_l \in I_0$ ($l = 1, 2, \ldots$), $\tau_i \neq \tau_j$ if $i \neq j$, and $\lim_{l \to +\infty} \tau_l = b_0$.

Let $H = \text{diag}(h_1, \ldots, h_n) : I_0 \to \mathbb{R}^{n \times n}$ be a diagonal matrix-functions with continuous diagonal elements $h_k : I_0 \to ]0, +\infty[ \; (k = 1, \ldots, n)$.

We consider the problem of the well-posedness of solution $x : I_0 \to \mathbb{R}^n$ of the system (1), (2), satisfying the modified Cauchy condition

$$
\lim_{t \to b_0} (H^{-1}(t)x(t)) = 0.
$$

(3)

Along with the system (1), (2) consider the perturbed singular system

$$
\frac{dx}{dt} = \tilde{P}(t)x + \tilde{q}(t) \text{ for a.a. } t \in I_0 \setminus \{ \tau_l \}_{l=1}^{+\infty},
$$

(4)

$$
x(\tau_l+) - x(\tau_l-) = \tilde{G}_l x(\tau_l) + \tilde{g}_l \quad (l = 1, 2, \ldots),
$$

(5)

where $\tilde{P} \in L_{loc}(I_0, \mathbb{R}^{n \times n})$, $\tilde{q} \in L_{loc}(I_0, \mathbb{R}^n)$, $\tilde{G}_l \in \mathbb{R}^{n \times n}$ ($l = 1, 2, \ldots$), $\tilde{g}_l \in \mathbb{R}^n$ ($l = 1, 2, \ldots$).

In the paper, we investigate the question when the unique solvability of the problem (1), (2); (3) guarantees the unique solvability of the problem (4), (5); (3) and also nearness of its solutions in the definite sense if matrix-functions $P$ and $\tilde{P}$, $G_l$ and $\tilde{G}_l$ ($l = 1, 2, \ldots$), and vector-functions $q$ and $\tilde{q}$ and $g_l$ and $\tilde{g}_l$ ($l = 1, 2, \ldots$) are accordingly close to each other.

The analogous problem for systems (1) of ordinary differential equations with singularities are investigated in [2–4].

The singularity of system (1) is considered in the sense that the matrix $P$ and vector $q$ functions, in general, are not integrable at the point $b$. In general, the solution of the problem (1), (2); (3)
is not continuous at the point $b$ and, therefore, it is not a solution in the classical sense. But its restriction on every interval from $I_0$ is a solution of the system (1), (2). In connection with this we give the example from [4].

Let $\alpha > 0$ and $\varepsilon \in ]0, \alpha[$. Then the problem

$$
\frac{dx}{dt} = -\frac{\alpha x}{t} + \varepsilon |t|^{\varepsilon - 1 - \alpha}, \quad \lim_{t \to 0}(t^{\alpha}x(t)) = 0
$$

has the unique solution $x(t) = |t|^{\varepsilon - \alpha} \text{sgn} t$. This function is not solution of the equation on the set $I = \mathbb{R}$, but its restrictions on $]-\infty, 0[$ and $]0, +\infty[$ are solutions of that.

We give sufficient conditions guaranteeing the well-posedness of the problem (1), (2); (3). The analogous results belong to I. Kiguradze [3, 4] for the modified Cauchy problem for systems of ordinary differential equations with singularities.

Some boundary value problems for linear impulsive systems with singularities are investigated in [1] (see, also references therein).

In the paper, the use will be made of the following notation and definitions.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R} = ]-\infty, +\infty[,$  $\mathbb{R}_+ = [0, +\infty[,$  $[a, b]$ and $]a, b[ (a, b \in \mathbb{R})$ are, respectively, closed and open intervals.

$\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max_{j=1,\ldots,m} \sum_{i=1}^n |x_{ij}|$.

If $X = (x_{ij})_{i,j=1}^{n,m} \in \mathbb{R}^{n \times m}$, then $|X| = \{(x_{ij})_{i,j=1}^{n,m} | i,j \geq 0 (i = 1, \ldots, n; j = 1, \ldots, m)\}$.

$\mathbb{R}_+^{n \times 1}$ is the space of all real column $n$-vectors $x = (x_i)_{i=1}^n$.

If $X \in \mathbb{R}^{n \times n}$, then $X^{-1}, \det X$ and $r(X)$ are, respectively, the matrix inverse to $X$, the determinant of $X$ and the spectral radius of $X$; $I_n$ is the identity $n \times n$-matrix.

The inequalities between the matrices are understood componentwise.

A matrix-function is said to be continuous, integrable, nondecreasing, etc., if each of its component is such.

$X(t-)\text{ and } X(t+)\text{ are, respectively, the left and the right limits of the matrix-function } X : [a, b] \to \mathbb{R}^{n \times m}\text{ at the point } t.$

$C([a, b]; D), \text{ where } D \subset \mathbb{R}^{n \times m}, \text{ is the set of all absolutely continuous matrix-functions } X : [a, b] \to D.$

$C_{loc}(I_0 \setminus \{\tau_i\}_{i=1}^{+\infty}, D)$ is the set of all matrix-functions $X : I_{t_0} \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I_0 \setminus \{\tau_i\}_{i=1}^{+\infty}$ belong to $\widetilde{C}([a, b]; D)$.

$L([a, b]; D)$ is the set of all integrable matrix-functions $X : [a, b] \to D$.

$L_{loc}(I_0; D)$ is the set of all matrix-functions $X : I_0 \to D$ whose restrictions to an arbitrary closed interval $[a, b]$ from $I_0$ belong to $L([a, b]; D)$.

A vector-function $x \in \widetilde{C}_{loc}(I_0 \setminus \{\tau_i\}_{i=1}^{+\infty}, \mathbb{R}^n)$ is said to be a solution of the system (1), (2) if

$$
x'(t) = P(t)x(t) + q(t) \text{ for a.a. } t \in I_{t_0} \setminus \{\tau_i\}_{i=1}^{+\infty}
$$

and there exist one-sided limits $x(\tau_i-)$ and $x(\tau_i+)$ ($l = 1, 2, \ldots$) such that the equalities (2) hold.

We assume that

$$
det(I_n + G_l) \neq 0 \text{ (} l = 1, 2, \ldots \text{).}
$$

The above inequalities guarantee the unique solvability of the Cauchy problem for the corresponding nonsingular systems, i.e. for the case when $P \in L_{loc}(I, \mathbb{R}^{n \times n})$ and $q \in L_{loc}(I, \mathbb{R}^n)$.

Let $\mathcal{N}_{t_0} = \{l \in \mathbb{N} : t \leq \tau_l < b\}$ and $I_0(\delta) = [b_0 - \delta, b_0] \cap I_0$ for every $\delta > 0.$
Definition. The problem (1), (2); (3) is said to be $H$-well-posed if it has the unique solution $x$ and for every $\varepsilon > 0$ there exists $\eta > 0$ such that the problem (4), (5); (3) has the unique solution $\tilde{x}$ and the estimate

$$\|H(t)(x(t) - \tilde{x}(t))\| < \varepsilon \text{ for } t \in I$$

holds for every $\tilde{P} \in L_{\text{loc}}(I_0, \mathbb{R}^{n \times n})$, $\tilde{q} \in L_{\text{loc}}(I_0, \mathbb{R}^n)$, $\tilde{G}_l \in \mathbb{R}^{n \times n}$ $(l = 1, 2, \ldots)$, $\tilde{g}_l \in \mathbb{R}^n$ $(l = 1, 2, \ldots)$ such that $\det(I_n + \tilde{G}_l) \neq 0$ $(l = 1, 2, \ldots)$,

$$\left\| \int_{t}^{b-} H^{-1}(s)\tilde{P}(s) - P(s)|H(s)\, ds \right\| + \left\| \sum_{l \in N_0} H^{-1}(\tau_l)|\tilde{G}_l - G_l|H(\tau_l)\right\| < \eta \text{ for } t \in I_0(\delta)$$

and

$$\left\| \int_{t}^{b-} H^{-1}(s)|\tilde{q}(s) - q(s)|\, ds \right\| + \left\| \sum_{l \in N_0} H^{-1}(\tau_l)|\tilde{g}_l - g_l|\right\| < \eta \text{ for } t \in I_0(\delta).$$

Let $P_0 \in L_{\text{loc}}(I_0, \mathbb{R}^{n \times n})$ and $G_{0l} \in \mathbb{R}^{n \times n}$ $(l = 1, 2, \ldots)$. Then a matrix-function $C_0 : I_0 \times I_0 \rightarrow \mathbb{R}^{n \times n}$ is said to be the Cauchy matrix of the homogeneous impulsive system

$$\frac{dx}{dt} = P_0(t)x, \quad (6)$$

$$x(\tau+) - x(\tau-) = G_{0l}x(\tau_l) \quad (l = 1, 2, \ldots), \quad (7)$$

if, for every interval $J \subset I_0$ and $\tau \in J$, the restriction of $C_0(\cdot, \tau) : I_0 \rightarrow \mathbb{R}^{n \times n}$ on $J$ is the fundamental matrix of the system (6), (7) satisfying the condition $C_0(\tau, \tau) = I_n$. Therefore, $C_0$ is the Cauchy matrix of (6), (7) if and only if the restriction of $C_0$ on $J \times J$, for every interval $J \subset I_0$, is the Cauchy matrix of the system in the sense of definition given in [5].

Theorem. Let there exist a matrix-function $P_0 \in L_{\text{loc}}(I_0, \mathbb{R}^{n \times n})$ and constant matrices $G_l \in \mathbb{R}^{n \times n}$ $(l = 1, 2, \ldots)$ and $B_0, B \in \mathbb{R}_{+}^{n \times n}$ such that

$$\det(I_n + G_{0l}) \neq 0 \quad (l = 1, 2, \ldots),$$

$$r(B) < 1,$$

and the estimates

$$|C_0(t, \tau)| \leq H(t)B_0 H^{-1}(\tau) \text{ for } b - \delta \leq t \leq \tau < b, \quad \tau \neq \tau_l \quad (l = 1, 2, \ldots),$$

$$|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}| \leq H(t)B_0 H^{-1}(\tau_l) \text{ for } b - \delta \leq t \leq \tau_l < b \quad (l = 1, 2, \ldots)$$

and

$$\left| \int_{t}^{b-} C_0(t, \tau)(P(\tau) - P_0(\tau))|H(\tau)\, d\tau \right| + \sum_{l \in N_0} |C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}| |G_l - G_{0l}|H(\tau_l) \leq H(t)B \text{ for } t \in I_0(\delta)$$

hold for some $\delta > 0$, where $C_0$ is the Cauchy matrix of the system (5), (6). Let, moreover,

$$\lim_{t \rightarrow b} \left( \left\| \int_{t_0}^{t} H^{-1}(\tau)|C_0(t, \tau)| |q(\tau)|\, d\tau \right\| + \left\| \sum_{l \in N_0} H^{-1}(\tau_l)|C_0(t, \tau_l)G_{0l}(I_n + G_{0l})^{-1}| |g_l|\right\| \right) = 0.$$

Then the problem (1), (2); (3) is $H$-well-posed.
References


