On Asymptotic Behavior of Solutions to Second-Order Regular and Singular Emden–Fowler Type Differential Equations with Negative Potential

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1 Introduction

Consider the second-order Emden–Fowler type differential equation

\[ y'' - p(x, y, y')|y|^k \text{sgn } y = 0, \quad k > 0, \quad k \neq 1, \]  

where the function \( p(x, u, v) \) defined on \( \mathbb{R} \times \mathbb{R}^2 \) is positive, continuous in \( x \), Lipschitz continuous in \( u, v \).

Asymptotic classification of all solutions to equation (1) in the case \( p = p(x) \) was described by I. T. Kiguradze and T. A. Chanturia in [13]. Asymptotic classification of non-extensible solutions to similar third- and fourth-order differential equations was obtained by I. V. Astashova (see [1,3–5]). Asymptotic classification of solutions to equation (1) for the bounded function \( p(x, u, v) \) is contained in [8,9].

Sufficient conditions providing \( \lim_{x \to a} |y'(x)| = +\infty, \ a \in \mathbb{R} \), were obtained in [13]. However, the question of separating two cases

\[ \lim_{x \to a} |y(x)| = +\infty \quad \text{and} \quad \lim_{x \to a} |y(x)| < +\infty \]  

remained open. The answer on this question for \( p(x, u, v) = \tilde{p}(x)|v|^{\lambda}, \ \lambda \neq 1 \) was considered in [11].

Asymptotic behavior of non-extensible solutions to equation (1) for unbounded function \( p(x, u, v) \) is investigated in [6,7,10]. By using methods described in [1,2], conditions on function \( p(x, u, v) \) and initial data providing the existence of a vertical asymptote to related solution (i.e. the first case of (2)) are obtained. Other conditions on \( p(x, u, v) \) and initial data sufficient for the second case of (2) are considered. Solutions satisfying the second condition of (2) are called \textit{black hole} solutions (see [12]).

2 Asymptotic classification of solutions to Emden–Fowler type differential equations with bounded negative potential

Let us use the notation

\[ \alpha = \frac{2}{k-1}, \quad C(\tilde{p}) = \left( \frac{\alpha + 1}{\tilde{p}} \right)^{\frac{1}{k}} = \left( \frac{\tilde{p}(k-1)^2}{2(k+1)} \right)^{\frac{1}{k}}. \]

\textbf{Definition 2.1.} A solution \( y(x) \) to (1) is called \textit{positive Kneser solution on} \((x_0; +\infty)\) if it satisfies the conditions \( y(x) > 0, \ y'(x) < 0 \) at \( x > x_0 \).
Definition 2.2. A solution $y(x)$ to (1) is called negative Kneser solution on $(x_0; +\infty)$ if it satisfies the conditions $y(x) < 0$, $y'(x) > 0$ at $x > x_0$.

Definition 2.3. A solution $y(x)$ to (1) is called positive Kneser solution on $(-\infty; x_0)$ if it satisfies the conditions $y(x) > 0$, $y'(x) > 0$ at $x < x_0$.

Definition 2.4. A solution $y(x)$ to (1) is called negative Kneser solution on $(-\infty; x_0)$ if it satisfies the conditions $y(x) < 0$, $y'(x) < 0$ at $x < x_0$.

Theorem 2.1. Suppose $k > 1$. Let the function $p(x, u, v)$ be continuous in $x$, Lipschitz continuous in $u$, $v$ and satisfying inequalities

$$0 < m \leq p(x, u, v) \leq M < +\infty.$$  \hspace{1cm} (3)

Let there also exist the following limits of $p(x, u, v)$:

1) $P_+$ as $x \to +\infty$, $u \to 0$, $v \to 0$,

2) $P_-$ as $x \to -\infty$, $u \to 0$, $v \to 0$,

and for any $c \in \mathbb{R}$,

3) $P_c^+$ as $x \to c$, $u \to +\infty$, $v \to \pm\infty$,

4) $P_c^-$ as $x \to c$, $u \to -\infty$, $v \to \pm\infty$.

Then all non-extensible solutions to (1) are divided into the following nine types according to their asymptotic behavior:

0. Defined on the whole axis trivial solution $y_0(x) \equiv 0$.

1–2. Defined on $(b, +\infty)$ positive and negative Kneser solutions with power asymptotic behavior near domain boundaries:

$$y_1(x) = C(P^+_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b + 0, \quad y_1(x) = C(P_+)(x - b)^{-\alpha}(1 + o(1)t), \quad x \to +\infty,$$

$$y_2(x) = -C(P^-_b)(x - b)^{-\alpha}(1 + o(1)t), \quad x \to b + 0, \quad y_2(x) = -C(P_-)(x - b)^{-\alpha}(1 + o(1)), \quad x \to +\infty.$$

3–4. Defined on $(-\infty, a)$ positive and negative Kneser solutions with power asymptotic behavior near domain boundaries:

$$y_3(x) = C(P^+_a)(a - x)^{-\alpha}(1 + o(1)), \quad x \to a - 0, \quad y_3(x) = C(P_+)(a - x)^{-\alpha}(1 + o(1)), \quad x \to -\infty,$$

$$y_4(x) = -C(P^-_a)(a - x)^{-\alpha}(1 + o(1)), \quad x \to a - 0, \quad y_4(x) = -C(P_-)(a - x)^{-\alpha}(1 + o(1)), \quad x \to -\infty.$$

5–6. Defined on $(a, b)$ positive and negative solutions with power asymptotic behavior near domain boundaries:

$$y_5(x) = C(P^+_a)(x - a)^{-\alpha}(1 + o(1)), \quad x \to a + 0, \quad y_5(x) = C(P^+_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b - 0,$$

$$y_6(x) = -C(P^-_a)(x - a)^{-\alpha}(1 + o(1)), \quad x \to a + 0, \quad y_6(x) = -C(P^-_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b - 0.$$

7–8. Defined on $(a, b)$ solutions with different signs and power asymptotic behavior near domain boundaries:

$$y_7(x) = C(P^+_a)(x - a)^{-\alpha}(1 + o(1)), \quad x \to a + 0, \quad y_7(x) = -C(P^-_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b - 0,$$

$$y_8(x) = -C(P^-_a)(x - a)^{-\alpha}(1 + o(1)), \quad x \to a + 0, \quad y_8(x) = C(P^+_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b - 0.$$
Definition 2.5 (see [5]). A solution \( y : (a, b) \to \mathbb{R} \) with \( -\infty \leq a < b \leq +\infty \) to any ordinary differential equation is called a MU-solution if the following conditions hold:

(i) the equation has no solution equal to \( y \) on some subinterval of \( (a, b) \) and not equal to \( y \) at some point of \( (a, b) \);

(ii) either there is no solution defined on another interval containing \( (a, b) \) and equal to \( y \) on \( (a, b) \) or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of \( (a, b) \).

Theorem 2.2. Suppose \( 0 < k < 1 \). Let the function \( p(x,u,v) \) be continuous in \( x \), Lipschitz continuous in \( u, v \) and satisfying inequalities (3). Let there also exist the following limits of \( p(x,u,v) \):

1) \( P_{++} \) as \( x \to +\infty, u \to +\infty, v \to +\infty \);
2) \( P_{+-} \) as \( x \to +\infty, u \to -\infty, v \to -\infty \);
3) \( P_{-+} \) as \( x \to -\infty, u \to +\infty, v \to -\infty \);
4) \( P_{--} \) as \( x \to -\infty, u \to -\infty, v \to +\infty \),

and for any \( c \in \mathbb{R} \) denote \( P_c = p(c,0,0) \).

Then all MU-solutions to equation (1) are divided into the following eight types according to their asymptotic behavior:

1-2. Defined on semi-axis \((b, +\infty)\) positive and negative solutions tending to zero with their derivatives as \( x \to b + 0 \) with power asymptotic behavior near domain boundaries:

\[
y_1(x) = C(P_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b + 0, \quad y_1(x) = C(P_{++})x^{-\alpha}(1 + o(1)), \quad x \to +\infty, \\
y_2(x) = -C(P_b)(x - b)^{-\alpha}(1 + o(1)), \quad x \to b + 0, \quad y_2(x) = -C(P_{+-})x^{-\alpha}(1 + o(1)), \quad x \to +\infty.
\]

3-4. Defined on semi-axis \((-\infty, a)\) positive and negative solutions tending to zero with their derivatives as \( x \to a - 0 \) with power asymptotic behavior near domain boundaries:

\[
y_3(x) = C(P_a)(a - x)^{-\alpha}(1 + o(1)), \quad x \to a - 0, \quad y_3(x) = C(P_{++})|x|^{-\alpha}(1 + o(1)), \quad x \to -\infty, \\
y_4(x) = -C(P_a)(a - x)^{-\alpha}(1 + o(1)), \quad x \to a - 0, \quad y_4(x) = -C(P_{+-})|x|^{-\alpha}(1 + o(1)), \quad x \to -\infty.
\]

5-6. Defined on the whole axis solutions with same signs and power asymptotic behavior near domain boundaries:

\[
y_5(x) = C(P_{++})x^{-\alpha}(1 + o(1)), \quad x \to +\infty, \quad y_5(x) = C(P_{+-})|x|^{-\alpha}(1 + o(1)), \quad x \to -\infty, \\
y_6(x) = -C(P_{++})x^{-\alpha}(1 + o(1)), \quad x \to +\infty, \quad y_6(x) = -C(P_{+-})|x|^{-\alpha}(1 + o(1)), \quad x \to -\infty.
\]

7-8. Defined on the whole axis solutions with different signs and power asymptotic behavior near domain boundaries:

\[
y_7(x) = C(P_{++})x^{-\alpha}(1 + o(1)), \quad x \to +\infty, \quad y_7(x) = -C(P_{+-})|x|^{-\alpha}(1 + o(1)), \quad x \to -\infty, \\
y_8(x) = -C(P_{++})x^{-\alpha}(1 + o(1)), \quad x \to +\infty, \quad y_8(x) = C(P_{+-})|x|^{-\alpha}(1 + o(1)), \quad x \to -\infty.
\]
3 Asymptotic behavior of solutions to Emden–Fowler type differential equations with unbounded negative potential

Lemma 3.1. Suppose $k > 1$. Let $p(x, u, v)$ be continuous in $x$, Lipschitz continuous in $u, v$, and bounded below by a positive constant. Let $y(x)$ be a nontrivial non-extensible solution to equation (1) satisfying the condition $y(x_0)y'(x_0) \geq 0$ or $y(x_0)y'(x_0) \leq 0$ at some point $x_0$. Then there exists $x^* \in (x_0, +\infty)$ or respectively $x_* \in (-\infty, x_0)$, such that

$$\lim_{x \to x^* - 0} |y'(x)| = +\infty \quad \text{or respectively} \quad \lim_{x \to x_* + 0} |y'(x)| = +\infty. \quad (4)$$

Lemma 3.2. Suppose $0 < k < 1$. Let $p(x, u, v)/|v|$ be continuous in $x$, Lipschitz continuous in $u, v$, for $v \neq 0$ and bounded below by a positive constant. Let $y(x)$ be a nontrivial non-extensible solution to equation (1) satisfying the condition $y(x_0)y'(x_0) \geq 0$ or $y(x_0)y'(x_0) \leq 0$ but not $y(x_0) = y'(x_0) = 0$ at some point $x_0$. Then there exists $x^* \in (x_0, +\infty)$ or respectively $x_* \in (-\infty, x_0)$ providing (4).

Using the substitutions $x \mapsto -x$, $y(x) \mapsto -y(x)$ we obtain an equation of the same type as (1). That is why we investigate asymptotic behavior of non-extensible positive solutions to equation (1) near the right domain boundary only.

Theorem 3.1. Suppose there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the function $p = p(x, u, v)$ has the representation $p = h(u)g(v)$, where the functions $h(u)$, $g(v)$ are continuous and bounded below by a positive constant, and for $0 < k < 1$ function $p$ additionally satisfies the conditions of Lemma 3.2. Then for any non-extensible solution $y(x)$ to equation (1) with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of (2) the line $x = x^*$ is a vertical asymptote if and only if

$$\int^{+\infty}_{v_0} \frac{v}{g(v)} \, dv = +\infty. \quad (5)$$

Theorem 3.2. Suppose for $k > 1$ or $0 < k < 1$ the function $p(x, u, v)$ satisfies the conditions of Lemma 3.1 or respectively Lemma 3.2. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \leq f(x, u)g(v)$ holds, where the function $f(x, u)$ is continuous, the function $g(v)$ is continuous, bounded below by a positive constant and satisfies the condition

$$\int^{+\infty}_{v_0} \frac{dv}{g(v)} = +\infty. \quad (6)$$

Then for any non-extensible solution $y(x)$ to equation (1) with initial data satisfying inequalities $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and with the first property of (2) the line $x = x^*$ is a vertical asymptote.

Theorem 3.3. Suppose for $k > 1$ or $0 < k < 1$ the function $p(x, u, v)$ satisfies the conditions of Lemma 3.1 or respectively Lemma 3.2. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \leq g(v)$ holds, where the function $g(v)$ is continuous and satisfies the condition (6). Then for any non-extensible solution $y(x)$ to equation (1) with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of (2) the line $x = x^*$ is a vertical asymptote.

Theorem 3.4. Suppose for $k > 1$ or $0 < k < 1$ the function $p(x, u, v)$ satisfies the conditions of Lemma 3.1 or respectively Lemma 3.2. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \geq g(v)$ holds, where the function $g(v)$ is continuous, bounded below
by a positive constant and doesn’t satisfy the condition (5). Then for any non-extensible solution $y(x)$ to equation (1) with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ and the first property of (2) we have

$$0 < \lim_{x \to x^*} y(x) < +\infty, \quad x^* - x_0 < \frac{1}{y^k(x_0)} \int_{y'(x_0)}^{+\infty} \frac{dv}{g(v)}.$$ 

**Theorem 3.5.** Suppose $k > 0$, $k \neq 1$. Let the function $p(x, u, v)$ be continuous in $x$, Lipschitz continuous in $u$, $v$. Let there exist constants $u_0 > 0$, $v_0 > 0$ such that for $u > u_0$, $v > v_0$ the inequality $p(x, u, v) \leq C|x|^{-\alpha}$ holds. Then any non-extensible solution $y(x)$ to equation (1) with initial data $y(x_0) \geq u_0$, $y'(x_0) \geq v_0$ can be extended to $(x_0, +\infty)$ and

$$\lim_{x \to +\infty} y(x) = \lim_{x \to +\infty} y(x) = +\infty.$$ 

**References**


