Oscillations of Solutions of Second Order Linear Differential Equations and the Corresponding Difference Equations

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The object of our investigation is establishing the conditions for oscillations of solutions of linear second order differential equations, provided the solutions of the corresponding difference equations oscillate. We also establish the converse result, namely, when the oscillation of the solutions of difference equations implies the oscillation of the solutions of the corresponding differential equations.

Consider the linear second order differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0. \tag{1}$$

The following equations are called the *functional difference equation* and the *difference equation*, corresponding to (1), respectively:

$$\triangle^2 x(t) + hp(t)\triangle x(t) + h^2 q(t)x(t) = 0, \tag{2}$$

$$\triangle_k^2 x(t_0) + hp(t_0+kh)\triangle_k x(t_0) + h^2 q(t_0+kh)x(t_0+kh) = 0. \tag{3}$$

Here

$$\begin{aligned}
\triangle x(t) &= x(t+h) - x(t),\\
\triangle^2 x(t) &= \triangle(\triangle x(t)) = x(t+2h) - 2x(t+h) + x(t),\\
\triangle_k x(t_0) &= x(t_0+(k+1)h) - x(t_0+kh),\\
\triangle_k^2 x(t_0) &= \triangle_k(\triangle_k x(t_0)).
\end{aligned}$$

Denote $x_k^h = x(t_0+kh)$ to be the solution of the equation (3), with $t_k = t_0 + kh$.

Definition 1. We say that the solution x_k^h of the equation (3) *changes sign* at t_k, if either of the following conditions hold:

1) $x_k^h x_{k+1}^h < 0$;

2) $x_k^h = 0$, $x_{k-1}^h x_{k+1}^h < 0$.

Definition 2. A solution x_k^h of (3) is called *oscillatory* on some interval if it has at least two changes of signs on this interval.

We study the equation (2) under the conditions that ensure the continuity of its solutions. Thus, we have the usual concept of a zero for the solutions of (2), and the notion of oscillations of its solutions is essentially the same as for the solutions of (1).

Now we present the main results about the relation between the oscillation of solutions of the equations (1), (2), (3). These equations are equivalent to the following systems

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -p(t)y - q(t)x, \end{cases} \tag{4}$$

$$\begin{cases} x(t+h) = x(t) + hy(t), \\ y(t+h) = y(t) - h(p(t)y(t) + q(t)x(t)), \end{cases} \tag{5}$$

$$\begin{cases} x_{k+1}^h = x_k^h + hy_k^h, \\ y_{k+1}^h = y_k^h - h(p(t_0+kh)y_k^h + q(t_0+kh)y_k^h). \end{cases} \tag{6}$$

Therefore, the solutions of the system (5) are uniquely determined by the initial functions $x = \varphi(t), y = \psi(t)$, $t \in [0,h]$ which satisfy the coherence condition

$$\begin{cases} \varphi(h) = \varphi(0) + h\psi(0), \\ \psi(h) = \psi(0) - h(p(0)\psi(0) + q(0)\varphi(0)). \end{cases} \tag{7}$$

In what follows we assume that $\varphi, \psi \in C([0,h])$. The solutions of the system (6) are uniquely determined by the initial data

$$x_0^h(t_0) = x_0, \quad y_0^h(t_0) = y_0.$$

Theorem 1. *Let p and q in* (1) *be Lipschitz on* $[0,a]$. *Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ the assertion holds:*

If $x(t)$ is a solution of (1)*, which starts at $t_0 \in [0,h]$ and has at least three zeros on the interval $[t_0,a)$, then the corresponding solution of the difference equation* (3) *oscillates on* $[t_0,a]$.

Consider now the equation (2), or the equivalent system (5). The following result follows from Theorem 1 and Lemma 1.

Theorem 2. *Assume that p and q in* (2) *are Lipschitz on* $[0,a]$. *Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ the following statement holds:*

> *Every solution of the system* (5) *with the initial functions $\varphi, \psi \in C([0,h])$, which satisfy the condition* (7)*, has oscillatory first component on the* $(0,a)$*, provided that there exists $t_0 \in [0,h]$ such that the solution of the equation* (1) *with the initial data*
>
> $$x(t_0) = \varphi(t_0), \dot{x}(t_0) = \psi(t_0)$$
>
> *has at least three zeros on* (t_0,a).

Consider the equation

$$\ddot{x} + p(t)x = 0 \tag{8}$$

and the corresponding functional difference equation

$$\triangle^2 x(t) + h^2 p(t)x(t) = 0, \tag{9}$$

and the difference equation

$$\triangle_k^2 x(t_0) + h^2 p(t_0 + kh) x(t_0 + kh) = 0 \tag{10}$$

with p satisfying the Lipschitz condition on $[0, a]$. Let

$$m = \min_{t \in [0,a]} p(t), \quad M = \max_{t \in [0,a]} p(t).$$

Assume

$$m > 0 \ \text{ and } \ a > \frac{3\pi}{\sqrt{m}}\,. \tag{11}$$

Then if

$$a - \overline{h} > \frac{3\pi}{\sqrt{m}}\,, \tag{12}$$

all solutions of (8) with the initial data $t_0 \in [0, \bar{h}]$ have at least three zeros on the interval $[t_0, a)$.

Corollary 1. *Let p be Lipschitz on $[0, a]$, and the conditions (11) and (12) hold. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ all solutions of equation (10) with the initial data given at $t_0 \in [0, h]$, oscillate on the $[t_0, a)$.*

Corollary 2. *Let p be Lipschitz on $[0, a]$, and the conditions (11) and (12) hold. Then there exists $h_0 > 0$ such that for all $0 < h \le h_0$ every solution of the system*

$$\begin{cases} x(t + h) = x(t) + hy(t), \\ y(t + h) = y(t) - hp(t)x(t) \end{cases}$$

with the initial functions $\varphi, \psi \in C([0, h])$ satisfying the coherence condition

$$\begin{cases} \varphi(h) = \varphi(0) + h\psi(0), \\ \psi(h) = \psi(0) - hp(0)\varphi(0)), \end{cases}$$

has an oscillatory first component on the interval $(0, a)$.

Assume the following conditions hold:

$$p(t) \ge 0, \ \ t \in [0, a], \tag{13}$$

$$p(t) \ \text{ is Lipschitz on } [0, a]. \tag{14}$$

The difference equation, corresponding to (8), is

$$\triangle_k^2 x + h^2 p(kh) x(kh) = 0. \tag{15}$$

The following theorem describes the relation between the oscillations of solutions of (8) and (15).

Theorem 3. *Let $p(t)$ satisfy the conditions (13) and (14). Then there exists h_0 such that for all $0 < h \le h_0$ the assertion holds:*

> *If x_k^h is a solution of (15) which has at least three changes of sign on the interval $[0, a]$, then the corresponding solution of differential equation (8) oscillates on $[0, a]$.*