New Convergence Theorem for the Abstract Kurzweil–Stieltjes Integral

Giselle Antunes Monteiro, Umi Mahnuna Hanung, Milan Tvrdý

Institute of Mathematics, Academy of Sciences of the Czech Republic, Praha, Czech Republic
E-mail: gam@math.cas.cz; hanungum@ugm.ac.id; tvrdy@math.cas.cz

In theory of Riemann integral, the impact of the Bounded Convergence Theorem, also called *Arzelà* or *Arzelà–Osgood* or *Osgood Theorem*, is comparable to the importance of the Lebesgue Dominated Convergence Theorem in the theory of the Lebesgue integration. In this work we are concerned with the abstract Kurzweil–Stieltjes integral, that is, the Stieltjes type integral for functions with values in a Banach space introduced by Š. Schwabik in [6]. Our aim is to present the Bounded Convergence Theorem in this abstract setting.

To make our statement more precise we need to fix some notations.

In what follows, X is the Banach space and $L(X)$ is the Banach space of all bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in X, while $\|\cdot\|_\infty$ stands for the supremum norm. Furthermore, $BV([a,b],X)$ denotes the set of functions valued in X of bounded variation on $[a,b]$ and $G([a,b],X)$ denotes the set of regulated functions.

Throughout the paper by $\int\limits_a^b \mathrm{d}[F]\,g$ we understand the abstract Kurzweil–Stieltjes integral of $g:[a,b]\to X$ with respect to $F:[a,b]\to L(X)$ in the sense of [6].

Main Theorem (Bounded Convergence Theorem). *Let $g\in G([a,b],X)$, a sequence $\{g_n\}\subset G([a,b],X)$ and $K\in[0,\infty)$ be such that*

$$\lim_{n\to\infty} g_n(t)=g(t) \ \text{ for } \ t\in[a,b]$$

and

$$\|g_n\|_\infty\leq K<\infty \ \text{ for } \ n\in\mathbb{N}.$$

Then for any $F\in BV([a,b],L(X))$ and $n\in\mathbb{N}$ the integrals $\int\limits_a^b \mathrm{d}[F]\,g$, $\int\limits_a^b \mathrm{d}[F]\,g_n$ exist and

$$\lim_{n\to\infty}\int\limits_a^b \mathrm{d}[F]\,g_n=\int\limits_a^b \mathrm{d}[F]\,g.$$

In the case of real valued functions, the proof of such convergence result is based either on Arzelà's Lemma or on other sophisticated tools (cf. e.g. [2, Theorem II.19.3.14]) that cannot be extended to the case of Banach space-valued functions. Nevertheless, a paper by J. W. Lewin [3], in which an elementary proof of Bounded Convergence Theorem is given for the Riemann integral, offered some enlightenment to this topic.

Our approach is inspired by some of the ideas presented in [3] encompassing some new concepts that we will present below.

Let J be a bounded interval in $\mathbb{R}$. We say that a finite set $D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\}\subset J$ is a *generalized division of* J if $\alpha_0<\alpha_1<\dots<\alpha_{\nu(D)}$. The set of all generalized divisions of the interval J is denoted by $\mathcal{D}^*(J)$.

Let $f:[a,b]\to X$ and let J be an arbitrary subinterval of $[a,b]$. Then we define the variation of f on J by

$$\operatorname{var}_J f = \sup_{D\in\mathcal{D}^*(J)} \bigg\{ \sum_{j=1}^{\nu(D)} \big\| f(\alpha_j) - f(\alpha_{j-1}) \big\|_X \bigg\}.$$

If $\operatorname{var}_J f < \infty$, we say that f is of bounded variation on J and we write $f \in BV(J,X)$. Let us note that for intervals this definition coincides with that used by Gordon in [1]. Furthermore, it is easy to see that it coincides also with the usual (Jordan's) notion of the variation if J is a compact interval.

Making use of the variation on arbitrary intervals we introduce the variation over elementary sets. Recall that a bounded set $E \subset \mathbb{R}$ is an elementary set if it is a finite union of intervals. Moreover, given an elementary set E we can determine a collection of intervals $\{J_k : k = 1,\dots,m\}$ such that $E = \bigcup_{k=1}^m J_k$ and the union $J_k \cup J_\ell$ is not an interval whenever $k \neq \ell$. Such collection, called *minimal decomposition*, is uniquely determined and the intervals forming this collection are pairwise disjoint.

Definition. Given a function $f : [a,b] \to X$ and an elementary subset E of $[a,b]$, the variation of f over E is

$$\operatorname{var}(f,E) = \sum_{k=1}^m \operatorname{var}_{J_k} f,$$

where $\{J_k : k = 1,\dots,m\}$ is the minimal decomposition of E.

Now, we present an analogue to Lewin's lemma from [3]. In comparison with Lewin's original version, we replace the Lebesgue measure by the variation of a given function over elementary sets. For the proof, we needed to extend the Jordan's decomposition of functions of bounded variation (for the classical setting, see [2, Theorem I.7.1]) to the abstract setting.

Lemma. *Let $\{A_n\}$ be a sequence of subsets of $[a,b]$ such that $A_{n+1} \subseteq A_n$, $n \in \mathbb{N}$, and $\bigcap_n A_n = \emptyset$. Given $f \in BV([a,b],X)$, for $n \in \mathbb{N}$ put*

$$v_n = \sup\big\{ \operatorname{var}(f,E) \colon E \textit{ is an elementary subset of } A_n \big\}.$$

Then $\lim\limits_{n\to\infty} v_n = 0$.

In order to apply the previous lemma in the proof of our main result we need to introduce the notion of the Kurzweil–Stieltjes integral over elementary sets.

Definition. Let $F:[a,b]\to L(X)$, $g:[a,b]\to X$ and an elementary subset E of $[a,b]$ be given. The Kurzweil–Stieltjes integral of g with respect to F over E is given by

$$\int_E \mathrm{d}[F]g = \int_a^b \mathrm{d}[F](g\,\chi_E)$$

provided the integral on the right-hand side exists in the sense of [6].

Many basic properties of the integral defined above are immediate consequences of what is known for the abstract Kurzweil–Stieltjes integral, see [4] and [6]. Moreover, the integral over elementary sets in terms of its minimal decomposition can be calculated as follows.

Proposition. *Let $F \in BV([a,b],L(X))$, $g:[a,b]\to X$ and an elementary subset E of $[a,b]$ be such that the integral $\int_E \mathrm{d}[F]\,g$ exists. Then*

$$\int_E \mathrm{d}[F]\,g = \sum_{k=1}^m \int_{J_k} \mathrm{d}[F]\,g,$$

where $\{J_k\colon k = 1\dots,m\}$ *is the minimal decomposition of* E.

If we assume in addition that F *is continuous on* $[a,b]$, *then*

$$\bigg\| \int\limits_E \mathrm{d}[F]\, g \bigg\|_X \leq \operatorname{var}(F,E)\Big(\sup_{t\in E} \|g(t)\|_X\Big).$$

This proposition together with the analogue of Lewin's lemma mentioned above were the main tools that enabled us to prove the main result of this communication.

References

[1] R. A. Gordon, The integrals of Lebesgue, Denjoy, Perron, and Henstock. Graduate Studies in Mathematics, 4. *American Mathematical Society, Providence, RI*, 1994.

[2] T. H. Hildebrandt, Introduction to the theory of integration. Pure and Applied Mathematics, Vol. XIII. *Academic Press, New York–London*, 1963.

[3] J. W. Lewin, The teaching of mathematics: a truly elementary approach to the bounded convergence theorem. *Amer. Math. Monthly* **93** (1986), o. 5, 395–397.

[4] G. A. Monteiro and M. Tvrdý, On Kurzweil–Stieltjes integral in a Banach space. *Math. Bohem.* **137** (2012), No. 4, 365–381.

[5] Š. Schwabik, Generalized ordinary differential equations. Series in Real Analysis, 5. *World Scientific Publishing Co., Inc., River Edge, NJ*, 1992.

[6] Š. Schwabik, Abstract Perron–Stieltjes integral. *Math. Bohem.* **121** (1996), No. 4, 425–447.