Positive Solutions of Periodic Type Boundary Value Problems for Nonlinear Hyperbolic Equations with Singularities in the Phase Variable

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Let $a > 0$, $b > 0$,

$$D_{ab} = [0, a] \times [0, b], \quad D_b = \mathbb{R} \times [0, b].$$

In the rectangle D_{ab} and the strip D_b, respectively, consider the boundary value problems

$$u_{xy} = f(x, y, u), \tag{1}$$

$$u(0, y) = \lambda_1 u(a, y), \quad u(x, 0) = \lambda_2 u(x, b) \tag{2}$$

and

$$u_{xy} = p(x)u_y + q(x, y, u), \tag{3}$$

$$u(x + a, y) = u(x, y), \quad u(x, 0) = \lambda u(x, b). \tag{4}$$

Here

$$0 < \lambda_i < 1 \ \ (i = 1, 2), \quad 0 < \lambda < 1,$$

and $f : D_{ab} \times (0, +\infty) \to [0, +\infty)$, $p : \mathbb{R} \to \mathbb{R}$ and $q : D_b \to [0, +\infty)$ are continuous functions. Furthermore,

$$p(x + a) = p(x), \quad q(x + a, y, z) = q(x, y, z) \ \text{ for } \ (x, y) \in D_b, \ \ z > 0.$$

A function $u : D_{ab} \to (0, +\infty)$ ($u : D_b \to (0, +\infty)$ is called a *positive solution* of equation (1) (equation (3)) if it has continuous partial derivatives u_x, u_y, u_{xy} and satisfies equation (1) (equation (3)) in the rectangle D_{ab} (in the strip D_b).

A positive solution of equation (1) (equation (3)) satisfying the boundary conditions (2) (boundary conditions (4)) is called a *positive solution* of problem (1), (2) (problem (3), (4)).

The existence theorems formulated below cover the case where

$$\lim_{z \to 0} f(x, y, z) = +\infty, \quad \lim_{z \to 0} q(x, y, z) = +\infty,$$

i.e. the case, where equations (1) and (3) are singular with respect to the phase variable.

Similar results for ordinary differential equations are established in [1].

Introduce the functions

$$g_1(x, s) = \begin{cases} \dfrac{1}{1 - \lambda_1} & \text{for } \ 0 \le s \le x \le a \\ \dfrac{\lambda_1}{1 - \lambda_1} & \text{for } \ 0 \le x < s \le a \end{cases},$$

$$g_2(y, t) = \begin{cases} \dfrac{1}{1 - \lambda_2} & \text{for } \ 0 \le t \le y \le b \\ \dfrac{\lambda_2}{1 - \lambda_2} & \text{for } \ 0 \le y < t \le ba \end{cases}.$$

Theorem 1. *Let the inequality*

$$h_0(x,y,z) \le f(x,y,z) \le h_1(x,y,z)\Big(1 + \frac{z}{v(x,y)}\Big)$$

hold on the set $D_{ab} \times (0,+\infty)$*, where* $h_i : D_{ab} \times (0,+\infty) \to [0,+\infty)$ $(i = 0,1)$ *are continuous functions nonincreasing with respect to the third variable, and* $v : D_{ab} \to (0,+\infty)$ *is a continuous function. Moreover, let*

$$\max\Big\{h_0(x,y,z) : \ (x,y) \in D_{ab}\Big\} > 0 \ \text{ for } \ z > 0, \tag{5}$$

$$\lim_{z\to+\infty} h^*(z) < 1, \tag{6}$$

where

$$h^*(z) = \max\bigg\{\int\limits_0^a\int\limits_0^b \frac{g_1(x,s)g_2(y,t)}{v(x,y)}\, h_1(s,t,z)\, ds\, dt : \ (x,t) \in D_{ab}\bigg\}.$$

Then problem (1), (2) *has at least one positive solution.*

Corollary 1. *Let the inequality*

$$h_0(x,y,z) \le f(x,y,z) \le h_1(x,y,z)(1+z)$$

hold on the set $D_{ab} \times (0,+\infty)$*, where* $h_i : D_{ab} \times (0,+\infty) \to [0,+\infty)$ $(i = 0,1)$ *are continuous functions nonincreasing with respect to the third variable. Moreover, if* h_0 *satisfies condition* (5) *and* h_1 *satisfies the condition*

$$\lim_{z\to+\infty} \int\limits_0^a\int\limits_0^b h_1(x,y,z)\, dx\, dy < (1-\lambda_1)(1-\lambda_2),$$

then problem (1), (2) *has at least one positive solution.*

Corollary 2. *Let the inequality*

$$h_0(x,y,z) \le f(x,y,z) \le h_1(z)\Big(1 + \frac{z}{v_0(x,y)}\Big)$$

hold on the set $D_{ab} \times (0,+\infty)$*, where*

$$v_0(x,y) = \big((1-\lambda_1)x + \lambda_1 a\big)\big((1-\lambda_2)y + \lambda_2 b\big), \tag{7}$$

$h_0 : D_{ab} \times (0,+\infty) \to [0,+\infty)$ *is a continuous function nonincreasing with respect to the third variable and satisfying condition* (5)*, and* $h_1 : (0,+\infty) \to (0,+\infty)$ *is a nonincreasing continuous function such that*

$$\lim_{z\to+\infty} h_1(z) < (1-\lambda_1)(1-\lambda_2). \tag{8}$$

Then problem (1), (2) *has at least one positive solution.*

Corollary 3. *Let the inequality*

$$h_0(x,y,z) \le f(x,y,z) - \frac{l_0 z}{v_0(x,y)} \le h_1(x,y,z)(1+z)$$

hold on the set $D_{ab} \times (0,+\infty)$*, where* l_0 *is a nonnegative constant,* v_0 *is a function given by equality* (7)*, and* $h_i : D_{ab} \times (0,+\infty) \to [0,+\infty)$ $(i = 0,1)$ *are continuous functions nonincreasing with respect to the third variable. Furthermore, let* h_0 *satisfy condition* (5) *and let* h_1 *satisfy the condition*

$$\lim_{z\to+\infty} \int\limits_0^a\int\limits_0^b h_1(x,y,z)\, dx\, dy = 0. \tag{9}$$

Then problem (1), (2) *has at least one positive solution if and only if*

$$l_0 < (1-\lambda_1)(1-\lambda_2). \tag{10}$$

Example 1. Consider the equation

$$u_{xy} = \frac{l_0}{v_0(x,y)}\,u + \sum_{k=1}^{m} l_k(x,y)u^{-\mu_k}, \tag{11}$$

where l_0 is a nonnegative constant, $\mu_k > 0$ $(k = 1,\dots,m)$, and $l_k : D_{ab} \to (0,+\infty)$ $(k = 1,\dots,m)$ are continuous functions. According to Corollary 3, problem (11), (2) has at least one positive solution if and only if inequality (10) holds.

This example demonstrates that condition (6) (condition (8)) in Theorem 1 (in Corollary 2) is unimprovable and it cannot be replaced by a the nonstrict inequality

$$\lim_{z\to+\infty} h^*(z) \le 1 \quad \Big(\lim_{z\to+\infty} h_1(z) \le (1-\lambda_1)(1-\lambda_2)\Big).$$

Set

$$P_0(x) = \exp\bigg(\int_0^x p(s)\,ds\bigg), \ \ \lambda_0 = P_0(a). \tag{12}$$

On the basis of Corollaries 1–3 one can prove the following assertions on existence of a positive solution of problem (3), (4).

Corollary 4. *Let the inequality*

$$h_0(x,y,z) \le q(x,y,z) \le h_1(x,y,z)(P_0(x)+z),$$

hold on the set $D_{ab}\times(0,+\infty)$, *where* $h_i : D_{ab}\times(0,+\infty)\to[0,+\infty)$ $(i = 0,1)$ *are continuous functions nonincreasing with respect to the third variable. Furthermore, if*

$$\lambda_0 < 1,$$

h_0 *satisfies condition* (5) *and* h_1 *satisfies the condition*

$$\lim_{z\to+\infty} \int_0^a \int_0^b h_1(x,y,z)\,dx\,dy < (1-\lambda_0)(1-\lambda),$$

then problem (3), (4) *has at least one positive solution.*

Corollary 5. *Let* $\lambda_0 < 1$ *and let the inequality*

$$h_0(x,y,z) \le q(x,y,z) \le h_1(z)\Big(P_0(x) + \frac{z}{w_0(x,y)}\Big)$$

hold on the set $D_{ab}\times(0,+\infty)$, *where*

$$w_0(x,y) = \big((1-\lambda_0)x + \lambda_1 a\big)\big((1-\lambda)y + \lambda b\big), \tag{13}$$

$h_0 : D_{ab}\times(0,+\infty)\to[0,+\infty)$ *is a continuous function nonincreasing with respect to the third variable and satisfying condition* (5), *and* $h_1 : (0,+\infty)\to(0,+\infty)$ *is a nonincreasing continuous function such that*

$$\lim_{z\to+\infty} h_1(z) < (1-\lambda_0)(1-\lambda).$$

Then problem (3), (4) *has at least one positive solution.*

Corollary 6. *Let* $\lambda_0 < 1$ *and let the inequality*

$$h_0(x,y,z) \le q(x,y,z) - \frac{l_0 z}{w_0(x,y)} \le h_1(x,y,z)(1+z)$$

hold on the set $D_{ab} \times (0,+\infty)$*, where* l_0 *is a nonnegative constant,* w_0 *is a function given by equality* (13)*, and* $h_i : D_{ab} \times (0,+\infty) \to [0,+\infty)$ $(i = 0,1)$ *are continuous functions nonincreasing with respect to the third variable and satisfying conditions* (5) *and* (9)*. Then problem* (3), (4) *has at least one positive solution if and only if*

$$l_0 < (1-\lambda_0)(1-\lambda). \tag{14}$$

Example 2. Let $p : \mathbb{R} \to \mathbb{R}$ be an a-periodic continuous function, and let λ_0, P_0 and w_0 be the number and functions given by (12) and (13). Furthermore, $\lambda_0 < 1$. Consider the equation

$$u_{xy} = p(x)u_y + \frac{l_0}{w_0(x,y)}\, u + \sum_{k=1}^{m} l_k(x,y)u^{-\mu_k}, \tag{15}$$

where l_0 is a nonnegative constant, $\mu_k > 0$ $(k = 1,\dots,m)$, and $l_k : D_b \to (0,+\infty)$ $(k = 1,\dots,m)$ are continuous functions a-periodic with respect to the first variable. According to Corollary 6, problem (15), (4) has at least one positive solution if and only if inequality (14) holds.

References

[1] I. Kiguradze and Z. Sokhadze, Positive solutions of periodic type boundary value problems for first order singular functional differential equations. *Georgian Math. J.* **21** (2014), No. 3, 303–311.