On the Largest Lyapunov Exponent of the Linear Differential System with Parameter-Multiplier

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Consider the n-dimensional ($n \geq 2$) linear system of differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{1}$$

with piecewise continuous on the half-line $t \geq 0$ coefficient matrix $A(\,\cdot\,) : [0, +\infty) \to \mathbb{R}^n$. Denote the class of all such systems by $\mathcal{M}_n^*$. We identify the system (1) and it's coefficient matrix and therefore write $A \in \mathcal{M}_n^*$. Along with (1) we consider the one-parameter family

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mu A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{2}$$

of linear differential systems with a scalar parameter-multiplier $\mu \in \mathbb{R}$. Denote by $\mathcal{K}_n^*$ class of families (2) generated by systems $A \in \mathcal{M}_n^*$. Fixing in the family (2) the value of parameter μ we obtain the linear differential system which we denote by $\langle \mu \rangle_A$. Denote by $\lambda_1(\mu A) \leq \cdots \leq \lambda_n(\mu A)$ the Lyapunov exponents [1, p. 34], [2, p. 63] of the system $\langle \mu \rangle_A$.

V. I. Zubov in [3, p. 408, Problem 1] set the following problem: find out how the Lyapunov exponents of the systems (1) and (2) are related. Emphasize that in [3] in the formulation of the problem it is not necessary that the coefficient matrix of (1) to be bounded. Therefore exponent $\lambda_i(\mu A)$, $i = 1, \dots, n$, can take improper values $-\infty$ and $+\infty$. Hence the function $\lambda_i(\mu A)$ of a variable $\mu \in \mathbb{R}$ is a mapping $\mathbb{R} \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty, +\infty\}$. We call $\lambda_i(\mu A)$ the i-th Lyapunov exponent of the family (2).

In other words the problem of Zubov can be formulated as: for every $i = 1, \dots, n$ give a complete description of the set $\mathcal{L}_i^n \stackrel{\text{def}}{=} \{\lambda_i(\mu A) : \mathbb{R} \to \overline{\mathbb{R}} \mid \ A \in \mathcal{M}_n^*\}$ of i-th Lyapunov exponents of the families from $\mathcal{K}_n^*$.

In this article the problem of Zubov is solved for the largest Lyapunov exponent $\lambda_n(\mu A)$ on the assumption that $\lambda_n(\mu A)$ is not identically equal to $+\infty$ on any of the half-lines.

Note that for families of linear differential systems

$$\mathrm{d}x/\mathrm{d}t = A(t, \mu)x, \quad x \in \mathbb{R}^n, \quad t \geq 0, \tag{3}$$

with continuous in the variables t, μ and bounded on the half-line $t \geq 0$ for every fixed $\mu \in \mathbb{R}$ coefficient matrix $A(t, \mu) : [0, +\infty) \times \mathbb{R} \to \operatorname{End} \mathbb{R}^n$, a similar problem is solved in [4]. It is proved that for every $i = 1, \dots, n$ function $\lambda(\,\cdot\,) : \mathbb{R} \to \mathbb{R}$ is the i-th Lyapunov exponent (considered as a function of $\mu \in \mathbb{R}$) of some family (3) if and only if $\lambda(\,\cdot\,)$ belongs to the Baire class $({}^*, G_\delta)$ and have an upper semicontinuous minorant. In the paper [4] it is proved that this result holds in a more general situation – for the Lyapunov exponents of families of morphisms of Millionshchikov bundles.

Despite the fact that the dependence on the parameter in the families (2) is linear, the description of the largest Lyapunov exponents of families from $\mathcal{K}_n^*$ is similar to the description of the largest Lyapunov exponents in the general case of families (3).

We consider $\overline{\mathbb{R}}$ with a natural (order) topology, so that $\overline{\mathbb{R}}$ is homeomorphic to the interval $[-1,1]$. Choose such a homeomorphism $\ell : \overline{\mathbb{R}} \to [-1,1]$ in a standard way:

$$\ell(x) = \begin{cases} \dfrac{x}{|x|+1}, & \text{if } x \in \mathbb{R}, \\ \operatorname{sgn}(x), & x = \pm\infty. \end{cases}$$

Since the mapping ℓ performs an order-preserving homeomorphism between $\overline{\mathbb{R}}$ and $[-1,1]$, we say that function $f : \mathbb{R} \to \overline{\mathbb{R}}$ belongs to the Baire class $\mathcal{K}$ if composition $\ell \circ f$ belongs to the class $\mathcal{K}$.

Recall that a real-valued function is referred to as a function of the class $({}^*, G_\delta)$ [5, p. 223–224] if, for each $r \in \mathbb{R}$, the preimage of the interval $[r, +\infty)$ under the mapping f is a G_δ-set.

The following theorem describes the largest Lyapunov exponents of the families from $\mathcal{K}_n^*$ from the viewpoint of the Baire clasification.

Theorem 1. *Function $\lambda_n(\mu A) : \mathbb{R} \to \overline{\mathbb{R}}$ of a variable μ belongs to the class $({}^*, G_\delta)$ for any system $A \in \mathcal{M}_n^*$.*

Using an inequality similar to the Lyapunov inequality we get

Lemma 1. *Suppose that for some $\mu_0 \neq 0$ the largest Lyapunov exponent of the system $\langle \mu_0 \rangle_A$ of a family* (2) *is non-positive* (*can be $-\infty$*). *Then the largest Lyapunov exponent of the system $\langle \mu \rangle_A$ is non-negative for any $\mu \in \mathbb{R}$ such that $\mu\mu_0 \le 0$.*

The following theorem shows that assertions of Theorem 1 and Lemma 1 give us a sharp description of the restriction on some half-line of the largest Lyapunov exponents of the families from $\mathcal{K}_n^*$.

Theorem 2. *For any non-negative on some half-line function $f(\,\cdot\,) : \mathbb{R} \to \overline{\mathbb{R}}$ of the class $({}^*, G_\delta)$, there exist a system $A \in \mathcal{M}_n^*$ such that the largest Lyapunov exponent* (*as a function of $\mu \in \mathbb{R}$*) *of the system $\langle \mu \rangle_A$, coincides with $f(\,\cdot\,)$ on this half-line and is identically zero on the other half-line.*

Using the Lemma 1 we get further description of the properties of the largest Lyapunov exponents of the families from $\mathcal{K}_n^*$.

Lemma 2. *Suppose that for some $\mu_0 \neq 0$ the largest Lyapunov exponent of the system $\langle \mu_0 \rangle_A$ of a family* (2) *is finite and equals $\lambda \in \mathbb{R}$. Then the largest Lyapunov exponent of the system $\langle \mu \rangle_A$ satisfies the inequality $\lambda_n(\mu A) \ge \lambda\mu/\mu_0$ for any $\mu \in \mathbb{R}$ such that $\mu\mu_0 \le 0$.*

Using Lemma 1, Theorem 1 and Lemma 2 with some additional considerations we obtain

Theorem 3. *Function $\lambda_n(\mu A) : \mathbb{R} \to \overline{\mathbb{R}}$ of the variable μ is non-negative on some half-line, vanishes at zero and belongs to a class $({}^*, G_\delta)$ for any system $A \in \mathcal{M}_n^*$. Moreover, suppose that $\lambda_n(\mu A)$ takes at least one finite value on that half-line. Then there exist such a real number $b \in \mathbb{R}$ that the inequality $\lambda_n(\mu A) \ge b\mu$ holds for all $\mu \in \mathbb{R}$.*

Theorem 3 shows that the largest Lyapunov exponent of each family from $\mathcal{K}_n^*$ is non-negative on some half-line, vanishes at zero, belongs to a class $({}^*, G_\delta)$ and satisfies alternative: 1) it exceeds some linear function $b\mu$, or 2) it identically equals $+\infty$ on some half-line. In the first case these conditions are sufficient as shows the following theorem.

Theorem 4. *For each non-negative on some half-line function $f(\,\cdot\,) : \mathbb{R} \to \overline{\mathbb{R}}$ which vanishes at zero, belongs to the class $({}^*, G_\delta)$ and satisfies the inequality $f(\mu) \ge b\mu$ for any $\mu \in \mathbb{R}$ and some fixed $b \in \mathbb{R}$, there exist such a system $A \in \mathcal{M}_n^*$ that the largest Lyapunov exponent* (*as a function of μ*) *of the system $\langle \mu \rangle_A$ coincides with $f(\,\cdot\,)$.*

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