Stability of Linear Impulsive Itô Equations with Delay

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Stability analysis of Itô equations with aftereffect attracts attention of many researchers (see e.g. [2, 3, 8, 10, 11]). On the other hand, stability of impulsive functional differential equations is popular in the literature as well (see e.g. [1, 4]). In [7] we considered these two classes of equations together using the framework based on Azbelev's W-method (see e.g. [2, 3, 5, 6]).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and an increasing, right-continuous family (a filtration) $(\mathcal{F}_t)_{t\geq 0}$ of complete σ-subalgebras of $\mathcal{F}$. By $\mathbb{E}$ we denote the expectation on this probability space. The scalar stochastic processes $\mathcal{B}_i$, $i = 1, 2, \ldots, m$ form the standard m-dimensional Brownian motion on $(\mathcal{F}_t)_{t\geq 0}$ (see e.g. [9]).

Below we use the following spaces:

- L_∞ consists of all scalar, essentially bounded functions on $[0, \infty)$ with the usual norm;
- $\overline{L}_q^n$ consists of all n-dimensional progressively measurable (with respect to the above stochastic basis) stochastic processes on $[0, \infty)$, whose trajectories are almost surely (a.s.) locally q-integrable if $1 \leq q < \infty$ and a.s. locally bounded if $q = \infty$;
- k^n contains all $\mathcal{F}_0$-measurable n-dimensional random variables, and we will put $k^1 = k$ in the sequel;
- D^n contains all n-dimensional stochastic processes on $[0, \infty)$, which could be represented as $x(t) = x(0) + \int\limits_0^t f_0(s)\, ds + \sum\limits_{i=1}^m \int\limits_0^t f_i(s)\, d\mathcal{B}_i(s)$, where $x(0) \in k^n$, $f_0 \in \overline{L}_1^n$, $f_i \in \overline{L}_2^n$ $(i = 1, 2, \ldots, m)$.

Consider the following scalar Itô equation with impulses

$$dx(t) = \Big[-a(t)x(t) + \sum_{k=1}^{m_0} b_k(t)x(h_k(t))\Big]dt + \sum_{i=1}^{m}\Big[\sum_{k=1}^{m_i} c_{ik}(t)x(g_{ik}(t))\Big]d\mathcal{B}_i(t) \quad (t \geq 0), \tag{1}$$

$$x(s) = \varphi(s) \quad (s < 0),$$

$$x(\mu_j) = A_j x(\mu_j - 0), \quad j = 1, 2, 3, \ldots, \quad \text{a.s.}, \tag{2}$$

where $x(t, \omega) \in R^1$; μ_j, A_j, $j = 1, 2, 3, \ldots$, are real numbers such that $0 = \mu_0 < \mu_1 < \mu_2 < \cdots$, $\lim\limits_{j\to\infty} \mu_j = \infty$; $a, b_k \in \overline{L}_1^1$ for $k = 1, 2, \ldots, m_0$, $c_{ij} \in \overline{L}_2^1$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m_i$; h_k, g_{ij} are Borel measurable functions on $[0, \infty)$ such that $h_k(t) \leq t$, $g_{ij}(t) \leq t$ $(t \in [0, \infty))$ a.s. for $k = 1, 2, \ldots, m_0$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m_i$; φ is a scalar $\mathcal{F}_0$-measurable stochastic process with a.s. essentially bounded trajectories.

We remark that under these assumptions Eq. (1)–(2) is a special case of the general stochastic functional differential equation considered in [5,6]. In particular, any $x(0) \in k$ gives rise to a unique (up to the $\mathbb{P}$-equivalence) solution of this equation. We denote this solution by $x_\varphi(t, x_0)$.

Definition 1. The trivial solution of Eq. (1)–(2) is called exponentially p-stable with respect to the initial conditions if $\mathbb{E}|x_\varphi(t,x_0)|^p \le \bar{c}(\mathbb{E}|x_0|^p + \operatorname{vrai\,sup}_{\nu<0} \mathbb{E}|\varphi(\nu)|^p)\exp\{-\beta t\}$ $(t \ge 0)$ for some positive constants $\bar{c}$, β and all x_0 and φ described above.

Theorem 1. *Suppose that there exist positive numbers A, ρ, σ, δ, α such that $|A_j| \le A$, $\rho \le \mu_{j+1} - \mu_j \le \sigma$ for $j = 1,2,\dots$, $a(t) \ge \alpha$ $(t \in [0,\infty))$ a.s., $|b_k(t)| \le \bar{b}_k(t)$ $(t \in [0,\infty))$ a.s., where $\bar{b}_k \in L_\infty$ a.s. $k = 1,2,\dots,m_0$, $|c_{ij}(t)| \le \bar{c}_{ij}(t)$ $(t \in [0,\infty))$ a.s., where $\bar{c}_{ij} \in L_\infty$ for $i = 1,2,\dots,m$, $j = 1,2,\dots,m_i$, $t - h_k(t) < \delta$, $t - g_{ij}(t) < \delta$ $(t \in [0,\infty))$ almost everywhere for $k = 1,2,\dots,m_0$, $i = 1,2,\dots,m$, $j = 1,2,\dots,m_i$ and*

$$\frac{\max\{1,A\}(1-\exp\{-\alpha\sigma\})}{\alpha(1-\exp\{-\alpha\rho\}A)} \operatorname*{vrai\,sup}_{t\ge 0}\left\{\sum_{k=1}^{m_0}\bar{b}_k(t)\right\} + \\ + \left[\frac{\max\{1,A^2\}(1-\exp\{-2\alpha\sigma\})}{2\alpha(1-\exp\{-2\alpha\rho\}A)}\right]^{1/2} c_p \operatorname*{vrai\,sup}_{t\ge 0}\left\{\sum_{i=1}^{m}\sum_{k=1}^{m_i}\bar{c}_{ik}(t)\right\} < 1, \quad (3)$$

where c_p is the positive constant from the Burkholder–Davis–Gandy inequalities (see e.g. [9]). Then the trivial solution of Eq. (1)–(2) is exponentially $2p$-stable with respect to the initial conditions.

Sketch of the Proof

The proof of the theorem is based on Azbelev's W-transform of Eq. (1)-(2) (see e. g. [2], [3], [5]), which uses the so-called 'reference equation'. Normally, it is an equation which already possesses the desired asymptotic properties, but which is simpler than the given equation. The W-method works if an integral operator, which results from the substitution of the solutions of the reference equations into the given equation, is invertible.

Following this idea, we observe first of all that in our case it is sufficient to show that under the assumptions of the theorem the following inequality holds

$$\left(\mathbb{E}|x_\varphi(t,x_0)|^{2p}\right)^{1/2p} \le N\exp\{-\beta t\}\left[\|x_0\|_{k_{2p}} + \left(\operatorname*{vrai\,sup}_{s<0}\mathbb{E}|\varphi(s)|^{2p}\right)^{1/2p}\right], \quad (4)$$

where N and β are positive numbers.

Eq. (1) can be rewritten as follows:

$$dx(t) = \left[-a(t)x(t) + \sum_{k=1}^{m_0} b_k(t)(S_{h_k}x)(t) + f_0(t)\right]dt + \sum_{i=1}^{m}\left[\sum_{k=1}^{m_i} c_{ik}(t)(S_{g_{ik}}x)(t) + f_i(t)\right]d\mathcal{B}_i(t) \quad (t \ge 0), \quad (5)$$

where

$$f_0(t) = \sum_{k=1}^{m_0} b_k(t)\varphi_{h_k}(t), \quad f_i(t) = \sum_{k=1}^{m_i} c_{ik}(t)\varphi_{g_{ik}}(t) \ \text{ for } \ i = 1,2,\dots,m,$$

$$(S_h x)(t) = \begin{cases} x(h(t)), & \text{if } h(t) \ge 0, \\ 0, & \text{if } h(t) < 0, \end{cases} \quad \varphi_h(t) = \begin{cases} 0, & \text{if } h(t) \ge 0, \\ \varphi(h(t)), & \text{if } h(t) < 0. \end{cases}$$

The operator S_h is a stochastic analogue of *the inner superposition operator* [2]. In the stochastic case, if $h(t)$ is progressively measurable with respect to the given stochastic basis and $h(t) \le t$ $(t \in [0,+\infty))$ a.s., then the operator S_h maps the space D^n into the space $\overline{L}_\infty^n$, and it is also Volterra [6].

Note that the solution $x_\varphi(t,x_0)$ of Eq. (2), (5) satisfies (4) if the solution $y(t)$ of the equation

$$dy(t) = \left[\big(-a(t)+\beta\big)y(t) + \sum_{k=1}^{m_0} b_k(t)\exp\big\{\beta(t-h_k(t))\big\}(S_{h_k}y)(t) + f_0(t)\exp\{\beta t\}\right]dt +$$

$$+\sum_{i=1}^{m}\Big[\sum_{k=1}^{m_i} c_{ik}(t)\exp\big\{\beta(t-g_{ik}(t))\big\}(S_{g_{ik}}y)(t)+f_i(t)\exp\{\beta t\}\Big]d\mathcal{B}_i(t)\ \ (t\geq 0), \tag{6}$$

$$y(\mu_j)=A_jy(\mu_j-0),\ \ j=1,2,3,\dots,\ \ \text{a.s.}, \tag{7}$$

where β is some positive number, satisfies the inequality

$$\sup_{t\geq 0}\big(\mathbb{E}|y(t)|^p\big)^{1/p}\leq N\Big[\|y(0)\|_{k_{2p}}+\Big(\operatorname*{vrai\,sup}_{s<0}\mathbb{E}|\varphi(s)|^{2p}\Big)^{1/2p}\Big],$$

for some positive N.

To see this, we observe that there is the one-to-one correspondence $x_\varphi(t,x_0)=\exp\{-\beta t\}y(t)$ with $x_0=y(0)$ between the solutions of Eq. (2), (5) and Eq. (6)–(7).

Consider the following reference equation

$$\begin{aligned} dy(t)&=\big[-a(t)+\overline{f}_0(t)\big]dt+\sum_{i=1}^{m}\overline{f}_i(t)d\mathcal{B}_i(t)\ \ (t\geq 0),\\ y(\mu_j)&=A_jy(\mu_j-0),\ \ j=1,2,3,\dots,\ \ \text{a.s.}, \end{aligned} \tag{8}$$

coupled with Eq. (7), under the assumptions that $\overline{f}_0\in\overline{L}_1$, $\overline{f}_i\in\overline{L}_2$ for $i=1,2,\dots,m$.

It can be shown (see e.g. [6]) that any $y(0)\in k$ gives rise to a unique solution (up to the P-equivalence) of Eq. (8). A direct calculation yields the explicit representation of this solution:

$$\begin{aligned} y(t)=\exp\Big\{-\int_0^t a(s)\,ds\Big\}y(0)+\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_j\overline{f}_0(s)\,ds+\\ +\sum_{i=1}^{m}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_j\overline{f}_i(s)\,d\mathcal{B}_i(s)\ \ (t\geq 0). \end{aligned}$$

Using this representation we can rewrite Eq. (6), (7) as follows:

$$\begin{aligned} y(t)=\exp\Big\{-\int_0^t a(s)\,ds\Big\}y(0)+\beta\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jy(s)\,ds+\\ +\sum_{k=1}^{m_0}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jb_k(s)\exp\big\{\beta(s-h_k(s))\big\}(S_{h_k}y)(s)\,ds+\\ +\sum_{i=1}^{m}\sum_{k=1}^{m_i}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_j\exp\big\{s-g_{ik}(s)\big\}\,d\mathcal{B}_i(s)+\\ +\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jf_0(s)\,ds+\sum_{i=1}^{m}\int_0^t\exp\Big\{-\int_s^t a(\tau)\,d\tau\Big\}\prod_{s<\mu_j\leq t}A_jf_i(s)\,d\mathcal{B}_i(s)\ \ (t\geq 0). \end{aligned}$$

A direct estimation procedure (see [7] for the details) ends up with the following inequality:

$$\sup_{t\geq 0}\big(\mathbb{E}|y(t)|^p\big)^{1/p}\leq\|y(0)\|_{k_{2p}}+K\sup_{t\geq 0}\big(\mathbb{E}|y(t)|^p\big)^{1/p}+d\Big(\operatorname*{vrai\,sup}_{s<0}\mathbb{E}|\varphi(s)|^{2p}\Big)^{1/2p},$$

where

$$K=\frac{\max\{1,A\}(1-\exp\{-\alpha\sigma\})}{\alpha(1-\exp\{-\alpha\rho\}A)}\Big[\beta+\exp\{\beta\delta\}\operatorname*{vrai\,sup}_{t\geq 0}\Big\{\sum_{k=1}^{m_0}|b_k(t)|\Big\}\Big]+$$

$$+\Big[\frac{\max\{1,A^2\}(1-\exp\{-2\alpha\sigma\})}{2\alpha(1-\exp\{-2\alpha\rho\}A)}\Big]^{1/2} c_p \exp\{\beta\delta\} \operatorname*{vrai\,sup}_{t\geq 0}\Big\{\sum_{i=1}^{m}\sum_{k=1}^{m_i}|c_{ik}(t)|\Big\},$$

$$d=\frac{\max\{1,A\}(1-\exp\{-\alpha\sigma\})}{\alpha(1-\exp\{-\alpha\rho\}A)}\,d_0+c_p\Big[\frac{\max\{1,A^2\}(1-\exp\{-2\alpha\sigma\})}{2\alpha(1-\exp\{-2\alpha\rho\}A)}\Big]^{1/2}\sum_{i=1}^{m} d_i.$$

By the assumptions of the theorem, there exists $\beta>0$ such that $K<1$. This completes the proof.

Consider now a vector version of Eq. (1)–(2), where we assume that $x(\omega,t)\in R^n$, $a(t)$ is an $n\times n$-matrix, the entries of which belong to the space $\overline{L}_1^1$; $b_k(t)$ ia an $n\times n$-matrix, the entries of which belong to the space $\overline{L}_1^1$ for $k=1,2,\dots,m_0$; $c_{ij}(t)$ is an $n\times n$-matrix, the entries of which belong to the space $\overline{L}_2^2$ for $i=1,2,\dots,m$, $j=1,\dots,m_i$; A_j is an $n\times n$-matrix with real entries. In addition, we assume that for some functions $\widehat{b}_k$, $k=1,2,\dots,m_0$, $\widehat{c}_{ij}$, $i=1,2,\dots,m$, $j=1,2,\dots,m_i$, belonging to L_∞, the following estimates hold true: $\|b_k(t)\|\leq\widehat{b}_k(t)$ $(t\in[0,+\infty))$ a.s. for $k=1,2,\dots,m_0$, $\|c_{ij}(t)\|\leq\widehat{c}_{ij}$ $(t\in[0,+\infty))$ a.s. for $i=1,2,\dots,m$, $j=1,2,\dots,m_i$.

Theorem 2. *Suppose that there exist positive numbers A, ρ, σ, δ, α such that $\|A_j\|\leq A$, $\rho\leq\mu_{j+1}-\mu_j\leq\sigma$ for $j=1,2,\dots$, $\Big\|\exp\Big\{-\int\limits_s^t a(\tau)\,d\tau\Big\}\Big\|\leq\exp\{-\alpha(t-s)\}$ $(0\leq s\leq t<\infty)$ a.s., $t-h_k(t)<\delta$, $t-g_{ij}(t)<\delta$ $(t\in[0,\infty))$ almost everywhere for $k=1,2,\dots,m_0$, $i=1,2,\dots,m$, $j=1,2,\dots,m_i$, and in addition, the estimate* (3) *is valid where $\overline{b}_k(t)$ and $\overline{c}_{ik}(t)$ are replaced by $\widehat{b}_k(t)$ and $\widehat{c}_{ik}(t)$, respectively. Then the trivial solution of the vector version of Eq.* (1)–(2) *is exponentially $2p$-stable w.r.t. the initial function.*

The proof of this result essentially coincides with the proof of Theorem 1 if one replaces absolute values with vector norms and $n\times n$-matrix norms, respectively.

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