On the Periodic Problem for the Nonlinear Telegraph Equation with a Boundary Condition of Poincare

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In a plane of independent variables x and t in the strip $\Omega := \{(x,t)\in\mathbb{R}^2 :\ 0<x<l,\ t\in\mathbb{R}\}$ consider the problem of finding a solution $U(x,t)$ of the telegraph equation with power nonlinearity of the form

$$L_\lambda U := U_{tt} - U_{xx} + 2aU_t + cU + \lambda|U|^\alpha U = F(x,t),\ \ (x,t)\in\Omega, \tag{1}$$

satisfying the Poincare homogeneous boundary condition

$$\gamma_1 U_x(0,t) + \gamma_2 U_t(0,t) + \gamma_3 U(0,t) = 0,\ \ t\in\mathbb{R}, \tag{2}$$

and the Dirichlet condition

$$U(l,t) = 0,\ \ t\in\mathbb{R}, \tag{3}$$

respectively, for $x=0$ and $x=l$, and also the periodicity condition with respect to variable t

$$U(x,t+T) = U(x,t),\ \ x\in[0,l],\ \ t\in\mathbb{R}, \tag{4}$$

with constant real coefficients a, c, γ_i, $i=1,2,3$, and parameter $\lambda\neq 0$, where $\gamma_1\gamma_2\neq 0$. Here $T := const > 0$, $\alpha := const > 0$; F is a given, while U is an unknown real T-periodic with respect to time functions.

Remark 1. Let $\Omega_T := \Omega\cap\{0<t<T\}$, $f := F\big|_{\overline{\Omega}_T}$. Easy to see that if $U\in C^2(\overline{\Omega})$ is a classical solution of the problem (1)–(4), then function $u := U\big|_{\overline{\Omega}_T}$ represents a classical solution of the following nonlocal problem

$$L_\lambda u = f(x,t),\ \ (x,t)\in\Omega_T, \tag{5}$$

$$\gamma_1 u_x(0,t) + \gamma_2 u_t(0,t) + \gamma_3 u(0,t) = 0,\ \ u(l,t) = 0,\ \ 0\le t\le T, \tag{6}$$

$$(B_0 u)(x) = 0,\ \ (B_0 u_t)(x) = 0,\ \ x\in[0,l], \tag{7}$$

where $(B_\beta w)(x) := w(x,0) - \exp(-\beta T)w(x,T)$, $\beta\in\mathbb{R}$, $x\in[0,l]$, and, vice versa, if $f\in C(\overline{\Omega}_T)$ and $u\in C^2(\overline{\Omega}_T)$ is a classical solution of the problem (5)–(7), then function $U\in C^2(\overline{\Omega})$, being T-periodic with respect to time continuation of function u from the domain Ω_T into the strip Ω, will be a classical solution of the problem (1)–(4), if $f(x,0) = f(x,T)$, $x\in[0,l]$.

Definition 1. Let $f\in C(\overline{\Omega}_T)$ be a given function. Let $\Gamma_1 : x=0,\ 0\le t\le T$, $\Gamma_2 : x=l$, $0\le t\le T$. Function u is called a strong generalized solution of the problem (5)–(7) of the class C, if $u\in C(\overline{\Omega}_T)$ and there exists the sequence of functions $u_n\in \overset{\circ}{C}{}^2(\overline{\Omega}_T,\Gamma_1,\Gamma_2) := \big\{w\in C^2(\overline{\Omega}_T) : (\gamma_1 w_x + \gamma_2 w_t + \gamma_3 w)\big|_{\Gamma_1} = 0,\ w\big|_{\Gamma_2} = 0\big\}$ such that $u_n\to u$ and $L_\lambda u_n\to f$ in the space $C(\overline{\Omega}_T)$, while $B_0 u_n\to 0$ and $B_0 u_{nt}\to 0$ as $n\to\infty$, respectively, in the spaces $C^1([0,l])$ and $C([0,l])$.

Remark 2. It is obvious that classical solution of the problem (5)–(7) from the space $C^2(\overline{\Omega}_T)$ is a strong generalized solution of this problem of the class C.

To the periodic problem for nonlinear hyperbolic equations with boundary conditions of Dirichlet or Roben there is devoted comprehensive literature (see, e.g., [1–11] and the bibliography therein). In the present work it is investigated the periodic with respect to time problem (5)–(7), when the direction of derivative in the boundary condition does not coincide with the direction of the normal. The periodic problem is reduced to the one nonlocal with respect to time problem for solution of which it is proved a priori estimate. For the theorem of existence it is used representations of solutions of problems of Cauchy, Goursat and Darboux in different parts of the domain under consideration. The question of uniqueness is also considered.

When the following conditions are fulfilled

$$\lambda > 0, \;\; a > 0, \;\; c > 0; \;\; \gamma_1\gamma_2 < 0, \;\; \gamma_3\gamma_2 > 0, \tag{8}$$

then for the strong generalized solution u of the problem (5)–(7) of the class C it is proved the following a priori estimate

$$\|u\|_{C(\overline{\Omega}_T)} \le c\|f\|_{C(\overline{\Omega}_T)} \tag{9}$$

with positive constant $c = c(a, c, \gamma_i, l, T)$, not depending on functions u and f.

Remark 3. Note that the question of solvability of the problem (5)–(7) is reduced to the question of obtaining uniform with respect to parameter $\tau \in [0,1]$ a priori estimate of the strong generalized solution of the following equation

$$v_{tt} - v_{xx} + \tau(c - a^2)v + \tau\lambda\exp(-\alpha at)|v|^\alpha v = \tau\exp(at)f(x,t), \;\; (x,t) \in \overline{\Omega}_T, \tag{10}$$

satisfying the boundary

$$\gamma_1 v_x(0,t) + \gamma_2 v_t(0,t) + (\gamma_3 - a\gamma_2)v(0,t) = 0, \;\; v(l,t) = 0, \;\; 0 \le t \le T, \tag{11}$$

and nonlocal conditions

$$(B_a v)(x) = 0, \;\; (B_a v_t)(x) = 0, \;\; x \in [0,l]. \tag{12}$$

For obtaining uniform with respect to τ priori estimate for the solution of the problem (10)–(12) it is sufficient instead of (8) to require fulfilment of the following more restrictive conditions

$$\lambda > 0, \;\; a > 0, \;\; c \ge a^2, \;\; \gamma_1\gamma_2 < 0, \;\; \gamma_3\gamma_2^{-1} \ge a. \tag{13}$$

The following theorem is valid.

Theorem. *Let conditions* (13) *be fulfilled and $f \in C(\overline{\Omega}_l)$. Then the problem* (5)–(7) *has at least one strong generalized solution u of the class C in the sense of Definition* 1 *which belongs to the space $C^1(\overline{\Omega}_l)$, and when $f \in C^1(\overline{\Omega}_l)$, this solution is classical.*

Note that the problem (5)–(7) can not have more than one strong generalized solution of the class C in the domain Ω_l, if there hold conditions (13) and

$$|a^2 - c| < \frac{1}{c_0}\,, \;\; 0 < \lambda < \lambda_0,$$

where $\lambda_0 := (1 - c_0|a^2 - c|)(c_0 M_0)^{-1}$, $M_0 := (1+\alpha)(2c_1\|f\|_{C(\overline{\Omega}_l)})^\alpha$, and c_0, c_1 are definite positive constants, depending on a, γ_i, l.

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