Existence of Singular Solutions for Second Order Singular Differential Equations

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We consider second order singly singular differential equations of the types

$$\big(p(t)|x'|^{\alpha}\big)' + q(t)x^{-\beta} = 0, \tag{E}$$

and

$$\big(p(t)|x'|^{-\alpha}\big)' + q(t)x^{\beta} = 0, \tag{F}$$

under the assumption that

(a) α and β are positive constants;

(b) $p, q : [0,\infty) \to (0,\infty)$ are continuous functions.

By a solution on an interval J of (E) or (F) we mean a function $x : J \to (0,\infty)$ which is continuously differentiable on J together with $p(t)|x'(t)|^{\alpha}$ or $p(t)|x'(t)|^{-\alpha}$ and satisfies (E) or (F) there. If J is an unbounded interval of the form $[t_0,\infty)$, then $x(t)$ is said to be a *proper solution.* If J is a bounded interval of the form $[t_0,T)$ and $x(t)$ cannot be extended to the right beyond T, then $x(t)$ is called a *singular solution at* T. In this paper our attention is focused on singular solutions of (E) and (F) which are decreasing in their intervals of existence.

As is easily seen any singular solution $x(t)$ at T of (E) or (F) on $[t_0,T)$ has one of the following asymptotic behaviors

$$\lim_{t\to T-0} x(t) = A, \quad \lim_{t\to T-0} x'(t) = 0, \quad \text{for some } A > 0; \tag{I}$$

$$\lim_{t\to T-0} x(t) = A, \quad \lim_{t\to T-0} x'(t) = -\infty, \quad \text{for some } A > 0; \tag{II}$$

$$\lim_{t\to T-0} x(t) = 0, \quad \lim_{t\to T-0} x'(t) = 0; \tag{III}$$

$$\lim_{t\to T-0} x(t) = 0, \quad \lim_{t\to T-0} x'(t) = -B, \quad \text{for some } B > 0; \tag{IV}$$

$$\lim_{t\to T-0} x(t) = 0, \quad \lim_{t\to T-0} x'(t) = -\infty. \tag{V}$$

A singular solution satisfying (I) or (II) is termed a *white hole solution* or a *black hole solution*, respectively, while a singular solution satisfying (III), (IV) or (V) is termed an *extinct solution at T of the first kind, of the second kind or of the third kind*, respectively. Notice that the notion of black hole and white hole solutions was introduced by the present authors in [2] and [3].

It can be shown that equation (E) has white hole solutions but not black hole ones, whereas equation (F) may have black hole solutions but not white hole ones.

Theorem 1. *Equation* (E) *always has white hole solutions. More precisely, for any given $T > 0$ and $A > 0$ there exists a decreasing solution $x(t)$ of* (E) *satisfying* (I)*.*

Theorem 2. *Equation* (F) *has black hole solutions if and only if $\alpha > 1$, in which case, for any given $T > 0$ and $A > 0$ there exists a decreasing solution $x(t)$ of* (F) *satisfying* (II)*.*

The situations in which (E) and (F) possess extinct solutions of the second kind can be completely characterized as follows.

Theorem 3. *Equation* (E) *has extinct solutions of the second kind if and only if $\beta < 1$, in which case, for any given $T > 0$ and $B > 0$ there exists a decreasing solution $x(t)$ of* (E) *satisfying* (IV)*.*

Theorem 4. *Equation* (F) *always has extinct solutions of the second kind. More precisely, for any given $T > 0$ and $B > 0$ there exists a decreasing solution $x(t)$ of* (F) *satisfying* (IV)*.*

All of the above four theorems are verified by solving the appropriate integral equations with the help of the Schauder fixed point therems in Banach spaces. For example, the integral equations to be solved in Theorem 1 and Theorem 4 are

$$x(t) = A + \int_t^T \bigg(\frac{1}{p(s)} \int_s^T q(r) x(r)^\beta \, dr \bigg)^{-\frac{1}{\alpha}} ds,$$

and

$$x(t) = \int_t^T \bigg[\frac{1}{p(s)} \bigg(p(T) B^{-\alpha} + \int_s^T q(r) x(r)^{-\beta} \, dr \bigg) \bigg]^{\frac{1}{\alpha}} ds,$$

respectively.

It remains to ask whether (E) and (F) possess extinct solutions of the first and/or the third kinds. One easily sees that (E) (or (F)) cannot admit extinct solutions of the third kind (or the first kind). Information about the existence of such extinct solutions is provided by the following theorems in which the concept of *regularly varying functions at finite points*, defined below, plays a crucial role

Definition.

(i) A measurable function $f : [0, \infty) \to (0, \infty)$ is said to be *regularly varying at infinity of index* ρ (in the sense of Karamata) if it satisfies

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \ \lambda > 0$$

(ii) Let $T > 0$ be a finite constant. A measurable function $f : [0, T) \to (0, \infty)$ is said to be *regularly varying of index ρ at T* if $f(T - \tau^{-1})$, as a function of τ, is regularly varying of index $-\rho$ at infinity in the sense of Karamata.

The definition and some basic properties of regularly varying functions in the sense of Karamata can be found in [1, 6]. See also [5]. The concept of regularly varying functions at finite points has recently been introduced by the present authors [4].

The totality of regularly varying functions of index ρ at T is denoted by $\mathrm{RV}_T(\rho)$.The symbol SV_T is often used for $\mathrm{RV}_T(0)$, in which case its members are called *slowly varying functions at T*. By definition a function $f(t) \in \mathrm{RV}_T(\rho)$ is expressed in the form

$$f(t) = (T - t)^\rho L(t), \quad t \in [t_0, T), \quad \text{for some } \ L \in \mathrm{SV}_T.$$

Note that any positive continuous function on $[a, \infty)$ is slowly varying at any point $T \in (a, \infty)$, that is, a member of SV_T for any $T > 0$.

It is an easy task to show that most of the basic properties of regularly varying functions at infinity can be carried over to regularly varying functions at finite points. For instance, the Karamata integration theorem is translated into the following propositon, which is also referred to as the Karamata integration theorem for regularly varying functions at finite points.

Proposition. *Let* $L \in \mathrm{SV}_T$.

(i) *If* $\rho < -1$, *then*

$$\int\limits_a^t (T-s)^\rho L(s)\,ds \sim -\frac{1}{\rho+1}(T-t)^{\rho+1}L(t), \ \ t \to T-0.$$

(ii) *If* $\rho > -1$, *then*

$$\int\limits_t^T (T-s)^\rho L(s)\,ds \sim \frac{1}{\rho+1}(T-t)^{\rho+1}L(t), \ \ t \to T-0.$$

(iii) *If* $\rho = -1$, *then*

$$l(t) = \int\limits_a^t \frac{L(s)}{T-s}\,ds \in \mathrm{SV}_T \ \text{ and } \ \lim_{t\to T-0}\frac{L(t)}{l(t)} = 0,$$

and if $L(t)/(T-t)$ *is integrable in a left neighborhood of* T, *then*

$$m(t) = \int\limits_t^T \frac{L(s)}{T-s}\,ds \in \mathrm{SV}_T \ \text{ and } \ \lim_{t\to T-0}\frac{L(t)}{m(t)} = 0.$$

Applying the Schauder–Tychonoff fixed point theorem in combination with the above proposition to solve the integral equations

$$x(t) = \int\limits_t^T \bigg(\frac{1}{p(s)}\int\limits_s^T q(r)x(r)^\beta\,dr\bigg)^{-\frac{1}{\alpha}} ds, \tag{IE}$$

$$x(t) = \int\limits_t^T \bigg(\frac{1}{p(s)}\int\limits_s^T q(r)x(r)^{-\beta}\,dr\bigg)^{\frac{1}{\alpha}} ds, \tag{IF}$$

we are able to find criteria for the existence of extinct solutions of the first and the third kinds for (E) and (F) belonging to $\mathrm{RV}_T(\rho)$ with positive ρ.

Theorem 5. *Assume that* $\beta < \min\{\alpha, 1\}$. *Then, for any given* $T > 0$, *equation* (E) *has an extinct solution* $x(t)$ *at* T *of the first kind which belongs to the class* $\mathrm{RV}_T(\rho)$ *with*

$$\rho = \frac{\alpha+1}{\alpha+\beta}$$

and enjoys the exact asymptotic behavior

$$x(t) \sim \Big[\frac{(T-t)^{\alpha+1}p(t)^{-1}q(t)}{\alpha(\rho-1)\rho^\alpha}\Big]^{\frac{1}{\alpha+\beta}}, \ \ t \to T-0.$$

Theorem 6. *Assume that $\alpha > \max\{\beta, 1\}$. Then, for any given $T > 0$, equation* (F) *has an extinct solution $x(t)$ at T of the third kind which belongs to the class* $\mathrm{RV}_T(\rho)$ *with*

$$\rho = \frac{\alpha - 1}{\alpha + \beta}$$

and enjoys the exact asymptotic behavior

$$x(t) \sim \Big[\frac{(T-t)^{\alpha-1} p(t) q(t)^{-1}}{(\alpha(1-\rho))^{-1} \rho^{\alpha}}\Big]^{\frac{1}{\alpha+\beta}}, \quad t \to T - 0.$$

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