Invariant Manifolds of a Certain Class of Differential Equations

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We have established sufficient conditions for the existence of invariant toroidal manifolds of a certain class of linear extensions of dynamical system on torus. A similar problem for a sufficiently wide class of impulsive differential equations with non-fixed impulses also have been investigated.

We consider a system of differential equations, defined in the direct product of a torus $\mathcal{T}_m$ and an Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \tag{1}$$

where $\varphi = \operatorname{coll}(\varphi_1, \dots, \varphi_m)$, $x = \operatorname{coll}(x_1, \dots, x_n)$, $A(\varphi)$ and $f(\varphi)$ are continuous 2π-periodic with respect to each of the variable φ_j, $j = 1, \dots, m$ matrix and vector functions, respectively, defined on the torus $\mathcal{T}_m$.

We assume that the vector function $a(\varphi)$ satisfies the Lipschitz's condition

$$\big\|a(\varphi) - a(\psi)\big\| \leq \mathcal{L}\|\varphi - \psi\|, \tag{2}$$

for each $\varphi, \psi \in \mathcal{T}_m$ and some constant $\alpha > 0$.

Let $\varphi_t(\varphi)$, $\varphi_0(\varphi) = \varphi$ be the solution of the first of equations (1).

Consider the system of equations

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)) \tag{3}$$

that depends on φ as a parameter.

By invariant toroidal manifold of system (1) we will understand the set $x = u(\varphi), u(\varphi) \in C(\mathcal{T}_m)$, where $u(\varphi)$ is such function that $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution of system of equations (3) for each $\varphi \in \mathcal{T}_m$.

Deep research regarding the existence of invariant toroidal manifolds of differential equations were made by A. M. Samoilenko and the results of these studies are summarized in the classical monograph [1]. The main approach to the study of toroidal manifolds of system of equations (1) is based on the concept of Green–Samoilenko function of the invariant tori problem introduced in [1].

Let $\Omega_\tau^t(\varphi), \Omega_\tau^\tau(\varphi) = E$ be a fundamental matrix of system (3) and $C(\varphi)$ be a matrix function from the space $C(\mathcal{T}_m)$.

Let

$$G_0(\tau, \varphi) = \begin{cases} \Omega_\tau^0(\varphi)C(\varphi_\tau(\varphi)), & \tau \leq 0, \\ -\Omega_\tau^0(\varphi)E - C(\varphi_\tau(\varphi)), & \tau > 0. \end{cases} \tag{4}$$

Function $G_0(\tau, \varphi)$ is called Green–Samoilenko function of the invariant tori problem (1) if the following estimate holds

$$\int\limits_{-\infty}^{\infty} \|G_0(\tau, \varphi)\|\, d\tau \leq K < \infty, \ \ \varphi \in \mathcal{T}_m. \tag{5}$$

If system of equations (1) has a Green–Samoilenko function, it's invariant toroidal manifold may be represented as

$$x = u(\varphi) = \int\limits_{-\infty}^{\infty} G_0(\tau, \varphi) f(\varphi_\tau(\varphi))\, d\tau, \ \ \varphi \in \mathcal{T}_m.$$

Consider two classes of differential equations for which a Green–Samoilenko functions exist, so the invariant toroidal sets exist as well.

Let the matrix $A(\varphi_\tau(\varphi))$ from the system (1) commutes with its integral (Lappo–Danilevsky case, see [2, Part 2, § 13] for details).

$$A(\varphi_t(\varphi)) \int_\tau^t A(\varphi_s(\varphi))\,ds = \int_\tau^t A(\varphi_s(\varphi))\,ds \cdot A(\varphi_t(\varphi)) \tag{6}$$

for $t \geq \tau$.

Then

$$\Omega_\tau^t(\varphi) = e^{\int_\tau^t A(\varphi_s(\varphi))\,ds}$$

is a fundamental matrix of homogeneous system (3) and the following theorem holds.

Theorem 1. *Suppose that for any $t \geq \tau \geq 0$ a matrix $A(\varphi_t(\varphi))$ commutes with its integral. If all the eigenvalues of matrix*

$$A_0 = \lim_{t\to\infty} \frac{1}{t} \int_\tau^t A(\varphi_s(\varphi))\,ds \tag{7}$$

are on the left half of the complex plane λ, then system (1) has an asymptotically stable invariant toroidal manifold $x = u(\varphi)$, $\varphi \in \mathcal{T}_m$, for any $f(\varphi) \in C(\mathcal{T}_m)$.

Consider the case where the matrix $A(\varphi)$ in system (1) is such that the largest eigenvalue of the matrix

$$\widehat{A}(\varphi) = \frac{1}{2}\big(A(\varphi) + A^T(\varphi)\big),$$

where $A^T(\varphi)$ is a matrix transposed to $A(\varphi)$, $\Lambda(\varphi)$ is negative in ω-limit points of any solution $\varphi_t(\varphi)$ of the first equation from (1).

Using the Vazhevsky inequality [2], we see that in this case the function

$$G_0(\tau,\varphi) = \begin{cases} \Omega_\tau^0(\varphi), & \tau \leq 0, \\ 0, & \tau > 0 \end{cases} \tag{8}$$

satisfies the estimate

$$\|G_0(\tau,\varphi)\| \leq K e^{-\gamma|\tau|}, \ \ \tau \in R,$$

and it is a Green–Samoilenko function of the invariant tori problem. Thus, the system of equations (1) has an asymptotically stable invariant toroidal manifold

$$x = u(\varphi) = \int_{-\infty}^0 \Omega_\tau^0(\varphi) f(\varphi_\tau(\varphi))\,d\tau, \ \ \varphi \in \mathcal{T}.$$

Finally, we will develop the conditions for the existence of invariant toroidal sets of impulsive system of differential equations that undergo impulsive perturbation at the moments when the phase point meets a given set in the phase space

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = A(\varphi)x + f(\varphi), \ \ \varphi \notin \Gamma, \qquad \Delta x\big|_{\varphi\in\Gamma} = B(\varphi)x + g(\varphi). \tag{9}$$

Suppose the set Γ is a smooth $(m-1)$-dimensional submanifold of the torus $\mathcal{T}_m$ dimension and is determined by the equation $\Phi(\varphi) = 0$, where $\phi(\varphi)$ is a continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ function.

Let $t_i(\varphi)$, $i \in Z$, be solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$, which are the moments of impulsive perturbations in system (9), and assume that uniformly with respect to $t \in R$ there exists a limit

$$\lim_{T\to\infty} \frac{i(t, t+T)}{T} = \rho, \tag{10}$$

where $i(a, b)$ is a number of points $t_i(\varphi)$ in the interval (a, b).

Theorem 2. *Let a matrix $A(\varphi)$ satisfy the Lappo–Danilevsky condition for any $t \geq \tau \geq 0$ and uniformly with respect to $t \in R$ the finite limit* (10) *exist.*

Then, if

$$\gamma + \rho \ln \alpha < 0, \tag{11}$$

$$\gamma = \max_j \operatorname{Re} \lambda_j(A_0), \quad \alpha^2 = \max_{\varphi\in\mathcal{T}_m} \max_j \lambda_j (E + B(\varphi))^T (E + B(\varphi)),$$

then system of equations (9) *has an asymptotically stable invariant toroidal set*

$$x = u(\varphi) = \int_{-\infty}^{0} G_0(\tau, \varphi) f(\varphi_\tau(\varphi))\, d\tau + \\ + \sum_{t_i(\varphi)<0} G_0(t_i(\varphi) + 0, \varphi) g(\varphi_{t_i(\varphi)}(\varphi)), \quad \varphi \in \mathcal{T}_m.$$

Note that the conditions of Theorem 2 can be weakened by requiring the inequality (11) to be fulfilled not on the whole surface of a torus, but only in ω-limit sets of all solutions of the first equation from (9).

References

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