The Relation Between the Existence of Bounded Solutions of Differential Equations and the Corresponding Difference Equations

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We study the connection between the existence of bounded (on real axis) solutions of differential equations and the corresponding difference equations. We obtain the conditions under which the existence of bounded solutions of differential equations implies the existence of bounded solution of difference equation, and vice versa.

Throughout this work, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^d$ is the Euclidian space of d – dimensional vectors, $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ is the set of integers. The euclidian norm in $\mathbb{R}^d$ is denoted through $|\cdot|$, and $\|\cdot\|$ is the matrix norm in the same space.

Consider the following system of differential equations

$$\frac{dx}{dt} = X(t,x), \tag{1}$$

$t \in \mathbb{R}$, $x \in D$ for $D \subset \mathbb{R}^d$, and the corresponding system of difference equations

$$x^h(t+h) = x^h(t) + hX\big(t, x^h(t)\big), \tag{2}$$

where $h > 0$ is the step of difference equation. We assume that the function $X(t,x)$ is continuously differentiable and bounded together with its partial derivatives, i.e. $\exists\, C > 0$ such that

$$|X(t,x)| + \Big|\frac{\partial X(t,x)}{\partial t}\Big| + \Big\|\frac{\partial X(t,x)}{\partial x}\Big\| \le C \tag{3}$$

for $t \in R$, $x \in D$, where $\frac{\partial X}{\partial x}$ is the corresponding Jacobi matrix.

In this paper we study the connection between the existence of globally bounded solutions of (1) and of the corresponding system (2).

Here are some necessary statements and definition used later.

Consider the system (2) for $t = t_0 + kh$, where t_0 is fixed. We have

$$x^h_{k+1} = x^h_k + hX(t_0 + kh, x^h_k), \tag{4}$$

where $k \in \mathbb{Z}$, $h > 0$, $x^h_k = x^h(t_0 + kh)$, $x^h(t_0) = x_0$. The following results are used throughout the work.

Lemma 1. *Let $x(t)$ and x_k^h be the solutions of* (1) *and* (4) *on the interval* $[t_0, t_0+T]$ *such that* $x(t_0) = x_0^h = x_0$, $x \in D$. *Then, if the inequality* (3) *holds, we have the estimate*

$$\big|x(t_0+kh) - x_k^h\big| < he^{CT}[1+KT], \tag{5}$$

for $kh < T$, *where* $K = C + C^2$.

Lemma 2. *If the inequality* (3) *holds, any solution of* (4) x_k^h *continuously depends on the initial data* x_0, *until it reaches the boundary of* D.

Definition 1. We say that a solution $x^h(t)$ of system (2), defined for $t \in \mathbb{R}$, is exponentially stable uniformly in t_0 if there exist $\delta > 0$, $N > 0$ and $\alpha > 0$ such that for any solution $y^h(t)$ of the system (2) such that $|y^h(t_0) - x^h(t_0)| < \delta$ for $t \ge t_0$ we have the inequality

$$\big|x^h(t) - y^h(t)\big| \le Ne^{-\alpha(t-t_0)}\big|x^h(t_0) - y^h(t_0)\big|, \tag{6}$$

where constants δ, N and α do not depend on t_0.

Consider the system (4) for $t_0 = 0$:

$$x_{k+1}^h = x_k^h + hX(kh, x_k^h), \tag{7}$$

Definition 2. A solution x_k^h of (7) is called *exponentially stable* uniformly in k_0 if it satisfies the conditions in 1 with t_0 replaced with k_0 and t replaced with kh.

Our main results are the following theorems.

Theorem 1. *Assume the following conditions hold:*

1) *The function $X(t,x)$ satisfies* (3).
2) *There exists $h_0 > 0$ such that the system* (2) *has a bounded on $\mathbb{R}$, exponentially stable* (*in the sense of Definition* 1) *solution $x_k^{h_0}$, which lies in the domain D together with its ρ – neighborhood for some $\rho > 0$.*
3) *Additionally,*

$$h_0 e^{C(\frac{\ln 4N}{\alpha}+1)}\Big[1 + (C+C^2)\Big(\frac{\ln 4N}{\alpha} + 1\Big)\Big] \le \frac{\delta}{8}\,, \tag{8}$$

$$\frac{3N\delta}{2} < \rho, \tag{9}$$

$$h_0 \le \frac{\rho}{4C}\,, \tag{10}$$

where N, δ and α are defined in (6) *and C is given by* (3).

Then for all h, $0 < h < h_0$, the system (2) *has a bounded solution on $\mathbb{R}$.*

We proceed with studying the conditions for the existence of a bounded solution of (1), given that (2) has such a solution for $t = kh_0$.

The following theorem holds.

Theorem 2. *Let the following conditions hold:*

1) *the function $X(t,x)$ satisfies the condition* 1) *of Theorem* 1*;*
2) $\exists\, h_0 > 0$ *such that the system* (7) *has a bounded on $\mathbb{Z}$, uniformly in k_0 exponentially stable solution which belongs to the domain D together with its ρ neighborhood.*

Then, if the inequalities (9)–(10) *hold, the system* (1) *has a bounded solution defined on* $\mathbb{R}$.

We now proceed with the study the conditions on the existence of a bounded on $\mathbb{R}$ solution of the system (2) under the assumption that the system (1) has such a solution.

Theorem 3. *Let the following conditions hold:*

1) *The function* $X(t,x)$ *satisfies the condition* 1) *of* 1*;*
2) *The system* (1) *has a bounded on* $\mathbb{R}$*, asymptotically stable uniformly in* $t_0 \in \mathbb{R}$ *solution* $x(t)$*, which lies in the domain* D *with some* ρ *– neighborhood.*

Then there exists h_0 *such that for all* $0 < h \le h_0$ *the system* (2) *has a bounded on* $\mathbb{R}$ *solution* $x^h(t)$*, and*

$$\sup_{t\in R} \left|x^h(t) - x(t)\right| \to 0, \quad h \to 0 \tag{11}$$

The next example shows that the asymptotic stability of a bounded solution $x(t)$ is essential, and without this assumption we can get a qualitatively different behavior of solutions of differential and difference equations.

Example 1. Consider the differential equation

$$\ddot{x} + x = 0 \tag{12}$$

with the general solution

$$x(t) = C_1 \cos t + C_2 \sin t.$$

Its solutions are bounded, stable, but not asymptotically stable. The corresponding differential equation has the form

$$x((k+2)h) - 2x((k+1)h) + 2x(kh) = 0, \tag{13}$$

with the general solution

$$x_k^h = C_1 2^{\frac{hk}{2}} \cos \frac{hk\pi}{4} + C_2 2^{\frac{hk}{2}} \sin \frac{hk\pi}{4}\,.$$

We see that for all steps $h > 0$, all solutions of this equation (except the trivial one) are unbounded.