On Existence of Quasi-Periodic Solutions to a Nonlinear Singular Higher-Order Differential Equation and Asymptotic Classification of Its Solutions for the Forth Order

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1 Introduction

The paper is devoted to the existence of oscillatory quasi-periodic, in some sense, solutions to the higher-order singular Emden–Fowler type differential equation

$$y^{(n)} + p_0\,|y|^k \operatorname{sgn} y = 0, \quad n > 2, \quad k \in \mathbb{R}, \quad 0 < k < 1, \quad p_0 \neq 0, \tag{1}$$

and to the asymptotic classification of solutions to this equation with $n = 4$.

A lot of results about the asymptotic behavior of solutions to (1) are described in detail in [1] and [4]. Results on the existence of solutions with special asymptotic behavior are contained in [2, 3, 5–8]. Results on asymptotic classification of solutions to (1) with $n = 3$, $k > 0$, $k \neq 1$, and $n = 4$, $k > 1$, are given in [4] and [9].

2 On Existence of Quasi-Periodic Oscillatory Solutions

Put

$$\alpha = \frac{n}{k-1}.$$

Theorem 2.1. *For any integer $n > 2$ and real positive $k < 1$ there exists a non-constant oscillatory periodic function h such that for any p_0 with $(-1)^n p_0 > 0$ and any real x^* the function*

$$y(x) = |p_0|^{-\frac{1}{k-1}} (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad -\infty < x < x^*, \tag{2}$$

is a solution to equation (1).

Remark. Note that the same result for equation (1) with $n \geq 2$ and $k > 1$ was obtained earlier in [6–8]. A result on the existence of a positive solution similar to (2) with positive periodic function h for $n = 12, 13, 14$ and $k > 1$ is proved in [5].

3 On Asymptotic Classification of Solutions to Emden–Fowler Singular Equations of the Forth Order

The asymptotic classification of all possible solutions to the forth-order Emden–Fowler type differential equations

$$y^{\mathrm{IV}}(x) + p_0|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1, \quad p_0 > 0, \tag{3}$$

and

$$y^{\mathrm{IV}}(x) - p_0|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1, \quad p_0 > 0, \tag{4}$$

is given.

3.1 Definitions and Preliminary Results

In the case of regular nonlinearity $k > 1$, only maximally extended solutions are considered because solutions can behave in a special way only near the boundaries of their domains. If $k < 1$, then special behavior can occur also near internal points of the domains. This is why a notion of *maximally uniquely extended (MUE)* solutions is introduced.

Definition. A solution $u : (a, b) \to \mathbb{R}$ with $-\infty \le a < b \le +\infty$ to any ordinary differential equation is called a *MUE-solution* if the following conditions hold:

(i) the equation has no solution equal to u on some subinterval of (a, b) and not equal to u at some point of (a, b);

(ii) either there is no solution defined on another interval containing (a, b) and equal to u on (a, b) or there exist at least two such solutions not equal to each other at points arbitrary close to the boundary of (a, b).

In this article all MUE-solutions to equation (3) and (4) are classified according to their behavior near the boundaries of their domains. All maximally extended solution can be classified through investigation of possible ways to join several MUE-solutions.

Consider the equation

$$y^{(n)} + p(x, y, y', \dots, y^{(n-1)})\, |y|^k \operatorname{sgn} y = 0, \quad n \ge 2, \quad k \in \mathbb{R}, \quad 0 < k < 1,$$

with positive $p(x, y_0, \dots, y_{n-1})$.

Note that, because of the condition $0 < k < 1$, the classical theorem on the uniqueness of solutions cannot be applied to Cauchy problems with $y(x_0) = 0$. Nevertheless, the following assertion holds (see [4, 7.3]).

Theorem 3.1. *Let the function $p(x, y_0, \dots, y_{n-1})$ be continuous in x and Lipschitz continuous in $y_0, \dots, y_{n-1}$. Then for any tuple of numbers $x_0, y_0^0, \dots, y_{n-1}^0$ with not all y_i^0 equal to zero, the corresponding Cauchy problem $y(x_0) = y_0^0, \dots, y^{(n-1)}(x_0) = y_{n-1}^0$ has a unique solution.*

Remark. While the uniqueness conditions hold, the property of continuous dependence of solution on initial data fulfils (see [10, V, Theorem 2.1]).

3.2 Main Results. Asymptotic classification of solutions to equations (3) and (4)

Theorem 3.2. *Suppose $0 < k < 1$ and $p_0 > 0$. Then all MUE-solutions to equation* (3) *are divided into the following three types according to their asymptotic behavior* (*see Figure* 1).

1. *Oscillatory solutions defined on semi-axes $(-\infty, b)$. The distance between their neighboring zeros infinitely increases near $-\infty$ and tends to zero near b. The solutions and their derivatives satisfy the relations $\lim\limits_{x\to b} y^{(j)}(x) = 0$ and $\varlimsup\limits_{x\to-\infty} |y^{(j)}(x)| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum the following estimates hold:*

$$C_1|x-b|^{-\frac{4}{k-1}} \le |y(x)| \le C_2|x-b|^{-\frac{4}{k-1}} \tag{5}$$

   with positive constants C_1 and C_2 depending only on k and p_0.

2. *Oscillatory solutions defined on semi-axes $(b, +\infty)$. The distance between their neighboring zeros tends to zero near b and infinitely increases near $+\infty$. The solutions and their derivatives satisfy the relations $\lim\limits_{x\to b} y^{(j)}(x) = 0$ and $\varlimsup\limits_{x\to+\infty} |y^{(j)}(x)| = \infty$ for $j = 0, 1, 2, 3$. At the points of local extremum estimates* (5) *hold with positive constants C_1 and C_2 depending only on k and p_0.*

3. *Oscillatory solutions defined on* $(-\infty, +\infty)$. *All their derivatives* $y^{(j)}$ *with* $j = 0, 1, 2, 3, 4$ *satisfy*

$$\overline{\lim_{x\to-\infty}}\,|y^{(j)}(x)| = \overline{\lim_{x\to+\infty}}\,|y^{(j)}(x)| = \infty.$$

At the points of local extremum the estimates

$$C_1|x|^{-\frac{4}{k-1}} \le |y(x)| \le C_2|x|^{-\frac{4}{k-1}} \tag{6}$$

hold near $-\infty$ *or* $+\infty$ *with positive constants* C_1 *and* C_2 *depending only on* k *and* p_0.

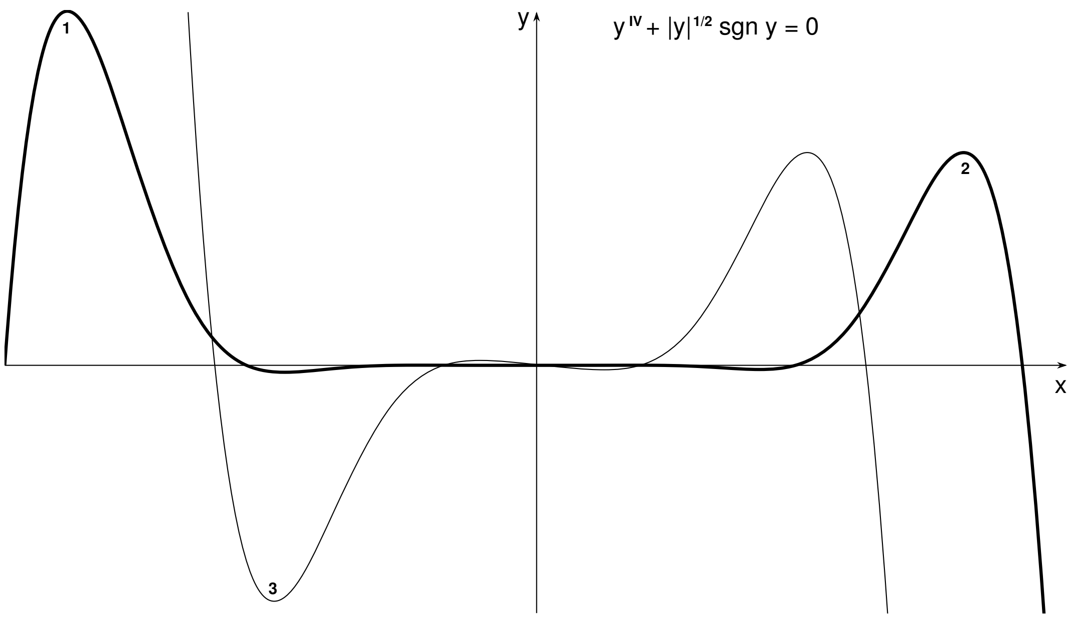

Figure 1.

Theorem 3.3. *Suppose* $0 < k < 1$ *and* $p_0 > 0$. *Then all MUE-solutions to equation* (4) *are divided into the following thirteen types according to their asymptotic behavior* (*see Figure* 2).

1–2. *Defined on semi-axes* $(b, +\infty)$ *solutions with the power asymptotic behavior near the boundaries of the domain* (*with the relative signs* $\pm$):

$$y(x) \sim \pm C_{4k}(x-b)^{-\frac{4}{k-1}}, \quad x \to b+0,$$
$$y(x) \sim \pm C_{4k}x^{-\frac{4}{k-1}}, \quad x \to +\infty,$$

where

$$C_{4k} = \Big(\frac{4(k+3)(2k+2)(3k+1)}{p_0\,(k-1)^4}\Big)^{\frac{1}{k-1}}.$$

3–4. *Defined on semi-axes* $(-\infty, b)$ *solutions with the power asymptotic behavior near the boundaries of the domain* (*with the relative signs* $\pm$):

$$y(x) \sim \pm C_{4k}|x|^{-\frac{4}{k-1}}, \quad x \to -\infty,$$
$$y(x) \sim \pm C_{4k}(b-x)^{-\frac{4}{k-1}}, \quad x \to b-0.$$

5. *Defined on the whole axis periodic oscillatory solutions. All of them can be received from one, say $z(x)$, by the relation*
$$y(x) = \lambda^4 z(\lambda^{k-1} x + x_0)$$
with arbitrary $\lambda > 0$ and x_0. So, there exists such a solution with any maximum $h > 0$ and with any period $T > 0$, but not with any pair (h, T).

6–7. *Defined on $(-\infty, +\infty)$ solutions that are oscillatory as $x \to -\infty$ and have the power asymptotic behavior near $+\infty$ (with the relative signs $\pm$):*
$$y(x) \sim \pm C_{4k} x^{-\frac{4}{k-1}}, \quad x \to +\infty.$$
For each solution a finite limit of the absolute values of its local extrema exists as $x \to -\infty$.

8–9. *Defined on $(-\infty, +\infty)$ solutions that are oscillatory as $x \to +\infty$ and have the power asymptotic behavior near $-\infty$ (with the relative signs $\pm$):*
$$y(x) \sim \pm C_{4k} |x|^{-\frac{4}{k-1}}, \quad x \to -\infty.$$
For each solution a finite limit of the absolute values of its local extrema exists as $x \to +\infty$.

10–13. *Defined on $(-\infty, +\infty)$ solutions having the power asymptotic behavior both near $-\infty$ and $+\infty$ (with the relative pairs of signs $\pm$):*
$$y(x) \sim \pm C_{4k}(p(b)) |x|^{-\frac{4}{k-1}}, \quad x \to \pm\infty.$$

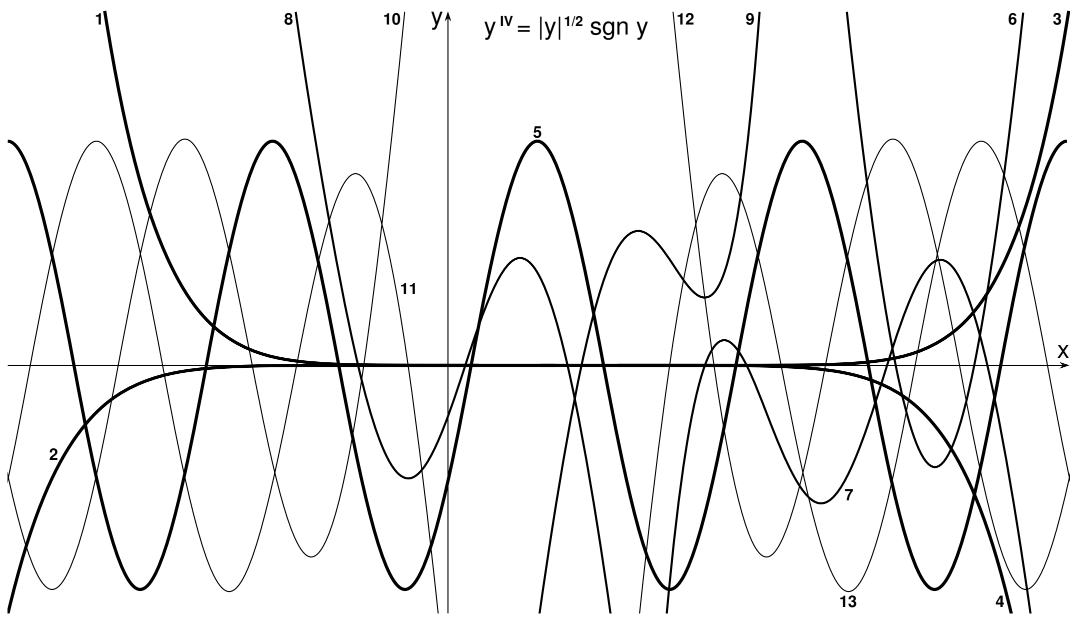

Figure 2.

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