Dynamics of Integro-Differential Cellular Neural Network Model

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1 Introduction

The main aim of this presentation is to study the dynamics of integro-differential Cellular Neural Network model from both theoretical and numerical points of view.

Let us first consider the following integro-differential problem:

$$\frac{\partial u}{\partial t} = D\,\frac{\partial^2 u}{\partial x^2} - \beta \int\limits_0^t e^{-\gamma(t-s)} u(s)\,ds + f(u), \quad t \in (0,T], \tag{1}$$

where γ and β are positive constants, f is a nonlinear function depending on u. If we take it in the form $f(u) = u(1-u)(u-\alpha)$, α being positive constant $0 < \alpha < 1$, this model describes the nerve impulse transmission and is known as FitzHugh–Nagumo equation [1]. For the equation (1) stability results are established in [1]. Finite difference method is proposed in order to solve (1) numerically. The application of this method to the above integro-differential problem needs a great storage of information in each time level. For this reason in this paper we shall propose Cellular Neural Networks (CNN) appraoch in order to study such kind of problems in real time due to the parallelism of the proposed architecture.

The model we shall consider is a more general form of the famous Hodgkin–Huxley model for propagation of the voltage pulse trough a nerve axon [3]:

$$u_t - D\nabla^2 u = \sigma u(u-\alpha)(1-u) - \beta \int^t g(u(s,x))\,ds, \tag{2}$$

where $0 < x,t < 1$, $0 < \alpha < 1$, σ, $\beta > 0$, D-the diffusion coefficients, g is a nonlinear function depending on u. The proposed equation (2) is a nonlinear parabolic integro-differential equation, $u(x,t)$ is a membrane in a nerve axon, the steady state $u = 0$ represents the resting state of the nerve. For (2) travelling wave solutions have been constructed in [5]. In this paper we shall study the dynamics of (2). We shall construct CNN architecture for integro-differential equation (2) in the nex section.

2 Integro-Differential CNN Model and its Dynamics

Cellular Neural Networks (CNNs) [2] are complex nonlinear dynamical systems, and therefore one can expect interesting phenomena like bifurcations and chaos to occur in such nets. It was shown that as the cell self-feedback coefficients are changed to a critical value, a CNN with opposite-sign template may change from stable to unstable. Namely speaking, this phenomenon arises as the loss of stability and the birth of a limit cycles.

It is known that some autonomous CNNs represent an excellent approximation to nonlinear partial differential equations (PDEs) [2]. The intrinsic space distributed topology makes the CNN able to produce real-time solutions of nonlinear PDEs. Consider the following well-known PDE, generally referred to us in the literature as a reaction-diffusion equation:

$$\frac{\partial u}{\partial t} = f(u) + D\nabla^2 u,$$

where $u \in \mathbf{R}^N$, $f \in \mathbf{R}^N$, D is a matrix with the diffusion coefficients, and $\nabla^2 u$ is the Laplacian operator in $\mathbf{R}^2$. There are several ways to approximate the Laplacian operator in discrete space by a CNN synaptic law with an appropriate A-template. An one-dimensional discretized Laplacian template will be in the following form:

$$A_1 = (1, -2, 1).$$

This is the two-dimensional discretized Laplacian A_2 template:

$$A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

For the integro-differential equation (2), CNN model will be the following:

$$\frac{du_{ij}}{dt} - DA_2 * u_{ij} = \sigma u_{ij}(1 - u_{ij})(u_{ij} - \alpha) - \beta \int_0^t g(u_{ij}(s))\,ds, \quad 1 \le i, j \le M. \tag{3}$$

We shall use an approximative method for studying the dynamics of integro-differential CNN (ID-CNN) model (3), based on a special Fourier transform. The idea of using Fourier expansion for finding the solutions of PDEs is well known in physics [4]. This special spectral technique is related to Harmonic Balance Method [4] well known in control theory and in the study of electronic oscillators as describing function method. The method is based on the fact that all cells in CNN are identical [2]. It is usually applied for discovering the existence and characteristics of periodic solutions.

In our case we apply the following double Fourier transform:

$$F(s, z_1, z_2) = \sum_{i=-\infty}^{\infty} z_1^{-i} \sum_{j=-\infty}^{\infty} z_2^{-j} \int_{-\infty}^{\infty} f_{ij}(t) \exp(-st)\,dt. \tag{4}$$

We apply this transform to ID-CNN model (3) and we obtain the following transfer function [4]:

$$H(s, z_1, z_2) = \frac{s}{s^2 - s(z_2^{-1} + z_2 - 4 + z_1^{-1} + z_1) + \beta}\,. \tag{5}$$

In the above transfer function $s = i\omega_0$, $z_1 \exp(i\Omega_1)$, $z_2 = \exp(i\Omega_2$, where ω_0, Ω_1, Ω_2 are temporal and two spatial frequencies, respectively, $i = \sqrt{-1}$.

We are looking for the periodic solutions of (3) of the form:

$$u_{ij}(t) = U_{m_0} \sin(\omega_0 t + i\Omega_1 + j\Omega_2), \tag{6}$$

where the temporal frequency is $\omega_0 = \frac{2\pi}{T_0}$, $T_0 > 0$ is the minimal period. If we take boundary conditions for ID-CNN model (3) which will make the array circular, we obtain:

$$\Omega_1 + \Omega_2 = \frac{2K\pi}{n}, \quad 0 \le K \le n-1, \quad n = M.M. \tag{7}$$

Applying describing function technique we obtain the following system for unknowns U_{m_0}, ω_0, Ω_1 and Ω_2:

$$\begin{gathered}\Omega_1 + \Omega_2 = \frac{2K\pi}{n}, \quad 0 \le K \le n-1, \\ 1 + \Big(\sigma\alpha + \frac{3}{4}\,\sigma U_{m_0}^2\Big)\,\frac{\omega_0 A}{(\beta-\omega_0^2)^2 + A\omega_0^2} = 0, \\ 1 + \Big(\sigma\alpha + \frac{3}{4}\,\sigma U_{m_0}^2\Big)\,\frac{\omega_0(\beta-\omega_0^2)}{(\beta-\omega_0^2)^2 + A\omega_0^2} = 0,\end{gathered} \tag{8}$$

where $A = 4 - 2\cos\Omega_1 - 2\Omega_2$.

Then the following proposition holds.

Proposition 1. *ID-CNN model* (3) *with circular array of M cells has periodic solutions with period $T_0 = \frac{2\pi}{\omega_0}$ and amplitude U_{m_0} for all $\Omega_1 + \Omega_2 = \frac{2K\pi}{n}$, $0 \le K \le n-1$.*

We obtain the following computer simulations of the solutions of ID-CNN model (3):

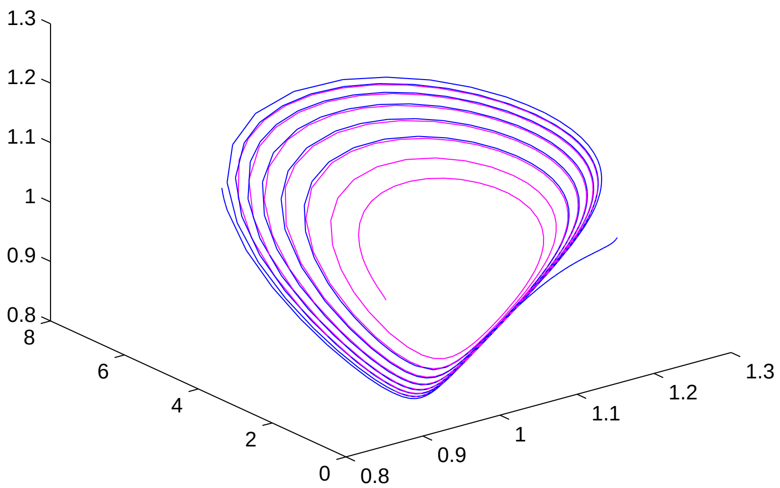

Figure 1. Computer simulations of the periodic solutions of ID-CNN model (3).

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