On Existence of Quasi-Periodic Solutions to a Nonlinear Higher-Order Differential Equation

I. Astashova
Lomonosov Moscow State University,
Moscow State University of Economics, Statistics and Informatics, Moscow, Russia
E-mail: ast@diffiety.ac.ru

1 Introduction
The paper is devoted to the existence of oscillatory and non-oscillatory quasi-periodic, in some sense, solutions to the higher-order Emden–Fowler type differential equation
\[ y^{(n)} + p_0 |y|^k \text{sgn} y = 0, \quad n > 2, \quad k \in \mathbb{R}, \quad k > 1, \quad p_0 \neq 0. \]  
(1)
A lot of results about the asymptotic behavior of solutions to (1) are described in detail in [1]. Results about the existence of solutions with special asymptotic behavior are contained in [2]–[8].

2 On Existence of Quasi-Periodic Oscillatory Solutions
Put
\[ \alpha = \frac{n}{k - 1}. \]
(2)

**Theorem 1.** For any integer \( n > 2 \) and real \( k > 1 \) there exists a non-constant periodic function \( h(s) \) such that for any \( p_0 > 0 \) and \( x^* \in \mathbb{R} \) the function
\[ y(x) = p_0^{\frac{1}{k-1}} (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad -\infty < x < x^* \]
(3)
is a solution to equation (1).

**Corollary 1.** For any integer even \( n > 2 \) and real \( k > 1 \) there exists a non-constant periodic function \( h(s) \) such that for any \( p_0 > 0 \) and \( x^* \in \mathbb{R} \) the function
\[ y(x) = p_0^{\frac{1}{k-1}} (x - x^*)^{-\alpha} h(\log(x - x^*)), \quad x^* < x < \infty \]
(4)
is a solution to equation (1).

**Corollary 2.** For any integer odd \( n > 2 \) and real \( k > 1 \) there exists a non-constant periodic function \( h(s) \) such that for any \( p_0 < 0 \) and \( x^* \in \mathbb{R} \) the function
\[ y(x) = |p_0|^{\frac{1}{k-1}} (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad x^* < x < \infty \]
(5)
is a solution to equation (1).

3 On Existence of Positive Solutions with Non-power Asymptotic Behavior
The existence of such non-oscillatory solutions was also proved.
For equation (1) with \( p_0 = -1 \) it was proved [4] that for any \( N \) and \( K > 1 \) there exist an integer \( n > N \) and \( k \in \mathbb{R} \) such that \( 1 < k < K \) and equation (1) has a solution of the form
\[ y = (x^* - x)^{-\alpha} h(\log(x^* - x)), \]
(6)
where \( \alpha \) is defined by (2) and \( h \) is a positive periodic non-constant function on \( \mathbb{R} \).
A similar result was also proved [4] about Kneser solutions, i.e. those satisfying \( y(x) \to 0 \) as \( x \to \infty \) and \( (-1)^j y^{(j)}(x) > 0 \) for \( 0 \leq j < n \). Namely, if \( p_0 = (-1)^{n-1} \), then for any \( N \) and \( K > 1 \) there exist an integer \( n > N \) and \( k \in \mathbb{R} \) such that \( 1 < k < K \) and equation (1) has a solution of the form
\[
y(x) = (x - x_*)^{-\alpha} h(\log(x - x_*)),
\]
where \( h \) is a positive periodic non-constant function on \( \mathbb{R} \).

Still it was not clear how large \( n \) should be for the existence of that type of positive solutions.

**Theorem 2** ([8]). If \( 12 \leq n \leq 14 \), then there exists \( k > 1 \) such that equation (1) with \( p_0 = -1 \) has a solution \( y(x) \) such that
\[
y^{(j)}(x) = (x^* - x)^{-\alpha - j} h_j(\log(x^* - x)), \quad j = 0, 1, \ldots, n - 1,
\]
where \( \alpha \) is defined by (2) and \( h_j \) are periodic positive non-constant functions on \( \mathbb{R} \).

**Remark.** Computer calculations give approximate values of \( \alpha \). They are, with the corresponding values of \( k \), as follows:
- if \( n = 12 \), then \( \alpha \approx 0.56, k \approx 22.4 \);
- if \( n = 13 \), then \( \alpha \approx 1.44, k \approx 10.0 \);
- if \( n = 14 \), then \( \alpha \approx 2.37, k \approx 6.9 \).

**Corollary 3** ([8]). If \( 12 \leq n \leq 14 \), then there exists \( k > 1 \) such that equation (1) with \( p_0 = (-1)^{n-1} \) has a Kneser solution \( y(x) \) satisfying
\[
y^{(j)}(x) = (x - x_0)^{-\alpha - j} h_j(\log(x - x_0)), \quad j = 0, 1, \ldots, n - 1,
\]
with periodic positive non-constant functions \( h_j \) on \( \mathbb{R} \).

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**References**


