On Invariant Sets of Impulsive Multi-Frequency Systems

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We consider a system of differential equations, defined in the direct product of an m-dimensional torus $\mathcal{T}_m$ and an n-dimensional Euclidean space E^n that undergo impulsive perturbations at the moments when the phase point φ meets a given set in the phase space

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= A(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x\big|_{\varphi\in\Gamma} &= B(\varphi)x + g(\varphi), \end{aligned} \tag{1}$$

where $\varphi = (\varphi_1, \dots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \dots, x_n)^T \in E^n$, $a(\varphi)$ is a continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ vector function that satisfies a Lipschitz condition with respect to φ. Functions $A(\varphi), B(\varphi)$ are continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ square matrices; $f(\varphi), g(\varphi)$ are continuous (piecewise continuous with first kind discontinuities in the set Γ) 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ vector functions.

We assume that the set Γ is a subset of the torus $\mathcal{T}_m$, which is a manifold of dimension $m-1$ defined by the equation $\Phi(\varphi) = 0$ for some continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ function.

Denote by $t_i(\varphi)$, $i \in \mathbb{Z}$ the solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$ that are the moments of impulsive action in system (1). Let the function $\Phi(\varphi)$ be such that the solutions $t = t_i(\varphi)$ exist since otherwise system (1) would not be an impulsive system.

We call a point φ^* an ω-limit point of the trajectory $\varphi_t(\varphi)$ if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that

$$\lim_{n\to+\infty} t_n = +\infty, \quad \lim_{n\to+\infty} \varphi_{t_n}(\varphi) = \varphi^*.$$

The set of all ω-limit points for a given trajectory $\varphi_t(\varphi)$ is called ω-limit set of the trajectory $\varphi_t(\varphi)$ and denoted by Ω_φ. Denote

$$\Omega = \bigcup_{\varphi\in\mathcal{T}_m} \Omega_\varphi,$$

and assume that the matrices $A(\varphi)$ and $B(\varphi)$ are constant in the domain Ω:

$$A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A}, \quad B(\varphi)\big|_{\varphi\in\Omega} = \widetilde{B}.$$

We will obtain sufficient conditions for the existence and asymptotic stability of an invariant set of the system (1) in terms of the eigenvalues of the matrices $\widetilde{A}$ and $\widetilde{B}$. Denote

$$\gamma = \max_{j=1,\dots,n} Re\lambda_j(\widetilde{A}), \quad \alpha^2 = \max_{j=1,\dots,n} \lambda_j\big((E+\widetilde{B})^T(E+\widetilde{B})\big).$$

Theorem 1. *Let the moments of impulsive perturbations $\{t_i(\varphi)\}$ be such that uniformly with respect to $t \in \mathbb{R}$ there exists a finite limit*

$$\lim_{\widetilde{T}\to\infty} \frac{i(t, t+\widetilde{T})}{\widetilde{T}} = p. \tag{2}$$

If the following inequality holds

$$\gamma + p \ln \alpha < 0, \tag{3}$$

then system (1) *has an asymptotically stable invariant set.*

Such approach may be extended to the nonlinear system of the form

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= A_0(\varphi)x + A_1(\varphi, x)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x\big|_{\varphi \in \Gamma} &= B_0(\varphi)x + B_1(\varphi, x)x + g(\varphi). \end{aligned} \tag{4}$$

Theorem 2. *Let the matrices $A_0(\varphi)$ and $B_0(\varphi)$ be constant in the domain Ω, uniformly with respect to $t \in \mathbb{R}$ there exist a finite limit* (2) *and the inequality* (3) *hold. Then there exist sufficiently small constants a_1 and b_1 and sufficiently small Lipschitz constants L_A and L_B such that for any continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ functions $A_1(\varphi, x)$ and $B_1(\varphi, x)$ such that $\max\limits_{\varphi \in \mathcal{T}_m, x \in \bar{J}_h} \|A_1(\varphi, x)\| \le a_1$, $\max\limits_{\varphi \in \mathcal{T}_m, x \in \bar{J}_h} \|B_1(\varphi, x)\| \le b_1$ and for any $x', x'' \in \bar{J}_h$,*

$$\|A_1(\varphi, x') - A_1(\varphi, x'')\| \le L_A \|x' - x''\|, \quad \|B_1(\varphi, x') - B_1(\varphi, x'')\| \le L_B \|x' - x''\|,$$

system (4) *has an asymptotically stable invariant set.*

In summary, we have obtained sufficient conditions for the existence and asymptotic stability of invariant sets of a linear impulsive system of differential equations defined in $\mathcal{T}_m \times E^n$ that has specific properties in the ω-limit set Ω of the trajectories $\varphi_t(\varphi)$. We have proved that it is sufficient to impose some restrictions on system (1) only in the domain Ω to guarantee the existence and asymptotic stability of the invariant set.

References

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