Generalized Linear Differential Equations in a Banach Space: Continuous Dependence on a Parameter

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In what follows, X is a Banach space and $L(X)$ is the Banach space of bounded linear operators on X. By $\|\cdot\|_X$ we denote the norm in a Banach space X. Further, $BV([a,b],X)$ is the set of X valued functions of bounded variation on $[a,b]$ and $G([a,b],X)$ is the set of X valued functions having on $[a,b]$ all one-sided limits (i.e. X valued functions regulated on $[a,b]$). A couple $P{=}(D,\xi)$, where $D{=}\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ and $\xi{=}(\xi_1,\dots,\xi_m){\in}[a,b]^m$, is said to be a partition of $[a,b]$ if $a{=}\alpha_0{<}\alpha_1{<}\dots{<}\alpha_m{=}b$ and $\alpha_{j-1}{\le}\xi_j{\le}\alpha_j$ for $j{=}1,2,\dots,m$. For such a partition P and functions $F{:}[a,b]{\to}L(X)$ and $g{:}[a,b]{\to}X$ we define

$$S(\mathrm{d}F,g,P){=}\sum_{j=1}^m\big[F(\alpha_j){-}F(\alpha_{j-1})\big]g(\xi_j)\ \text{ and }\ S(F,\mathrm{d}g,P){=}\sum_{j=1}^m F(\xi_j)\big[g(\alpha_j){-}g(\alpha_{j-1})\big].$$

For a gauge $\delta{:}[a,b]{\to}(0,\infty)$, the partition P is called δ-fine if

$$\big[\alpha_{j-1},\alpha_j\big]{\subset}\big(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j)\big)\ \text{ for all }\ j{\in}\mathbb{N}.$$

The integrals are the abstract *Kurzweil-Stieltjes integrals* (*KS-integrals*) defined as follows:

Definition. For $F{:}[a,b]{\to}L(X)$, $g{:}[a,b]{\to}X$ and $I{\in}X$ we say that $\int_a^b \mathrm{d}[F]g{=}I$ if for every $\varepsilon>0$ there exists a gauge δ on $[a,b]$ such that

$$\big\|S(\mathrm{d}F,g,P)-I\big\|_X<\varepsilon\ \text{ for all }\ \delta-\text{fine partitions } P \text{ of } [a,b].$$

Similarly we define the KS-integral $\int_a^b F\,\mathrm{d}[g]$ using sums of the form $S(F,\mathrm{d}g,P)$.

It is known that the integrals $\int_a^b \mathrm{d}[F]g$, $\int_a^b F\,\mathrm{d}[g]$ exist if $F{\in}G([a,b],L(X))$, $g{\in}G([a,b],X)$ and at least one of the functions F, g has a bounded variation on $[a,b]$ (cf. [2]). Further basic properties of the abstract KS-integral, like e.g. the substitution theorem, the integration-by-parts theorem or the convergence theorems, have been described in [6] and [2].

Let A, $A_k{\in}BV([a,b],L(X))$, $\widetilde{x},\widetilde{x}_k{\in}X$ and f, $f_k{\in}G([a,b],X)$ be given for $k{\in}\mathbb{N}$. Consider the generalized linear differential equations

$$x(t)=\widetilde{x}+\int_a^t \mathrm{d}[A(s)]x(s)+f(t)-f(a),\quad t\in[a,b], \tag{1}$$

and

$$x_k(t) = \widetilde{x}_k + \int_a^t \mathrm{d}[A_k(s)]x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b], \quad k \in \mathbb{N}. \tag{1_k}$$

The following assumptions are crucial for the existence of solutions to (1) and (1_k)

$$\big[I - \Delta^- A(t)\big]^{-1} \in L(X) \ \text{ for all } \ t \in (a,b], \tag{2}$$

and

$$\big[I - \Delta^- A_k(t)\big]^{-1} \in L(X) \ \text{ for all } \ t \in (a,b], \quad k \in \mathbb{N}. \tag{2_k}$$

For the basic properties of generalized linear differential equations in a Banach space, see [7].

Our first result extends that by M. Ashordia [1] valid for the case $X = \mathbb{R}^n$.

Theorem 1. *Let A, A_k satisfy* (1) *and* (1_k), *and let*

$$A_k \rightrightarrows A \ \text{ on } \ [a,b], \tag{3}$$

$$\alpha^* := \sup_{k \in \mathbb{N}} \big(\mathrm{v}\, ar_a^b A_k\big) < \infty, \tag{4}$$

$$f_k \rightrightarrows f \ \text{ on } \ [a,b], \tag{5}$$

$$\widetilde{x}_k \to \widetilde{x} \ \text{ in } \ X. \tag{6}$$

Then (1) *has a unique solution x on $[a,b]$. Furthermore, for each $k \in \mathbb{N}$ large enough there is a unique solution x_k on $[a,b]$ to* (1_k) *and $x_k \rightrightarrows x$.*

The next result extends that by Z. Opial [5] to homogeneous generalized linear differential equations in a general Banach space X.

Theorem 2. *Let $f(t) \equiv f(a), f_k(t) \equiv f_k(a)$ on $[a,b]$ for $k \in \mathbb{N}$ and let A, A_k satisfy* (1), (1_k). *Let $\widetilde{x}, \widetilde{x}_k \in X$ satisfy* (6) *and let*

$$\lim_{k \to \infty} \Big(\sup_{t \in [a,b]} \|A_k(t) - A(t)\|_{L(X)}\Big)\big(1 + \mathrm{v}\, ar_a^b A_k\big) = 0. \tag{6}$$

Then the conclusions of Theorem 1 *are true.*

For the proofs of Theorems 1 and 2, see [3]. The case when (3) (and hence also (6)) is not satisfied is treated in [4].

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