Asymptotic Equivalence of Linear Systems for Small Linear Perturbations

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1 Tests of Lyapunov's Reducibility of Linear Systems

Consider the linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in I = [0, +\infty), \tag{1}$$

with piecewise continuous and bounded (by the constant $a \geq \|A(t)\|$ for $t \in I$) coefficients. Along with systems (1), we will consider the systems

$$\dot{y} = (A(t) + Q(t))y, \quad y \in \mathbb{R}^n, \quad t \in I, \tag{2}$$

likewise with piecewise continuous and bounded on I coefficients.

Systems (1) and (2) are asymptotically equivalent (Lyapunov-equivalent, reducible) if there exists a linear transformation $x = L(t)y$, transferring one of the systems into another, where the matrix $L(t)$ is the Lyapunov one, i.e., satisfying the condition

$$\sup_{t \in I} \big\{ |L(t)\| + \|L^{-1}(t)\| + \|\dot{L}(t)\| \big\} < +\infty.$$

One of the tests of asymptotical equivalence of systems (1) and (2) is reflected [1] in the following assertion.

Theorem 1. *If* $\big\| \int\limits_t^{+\infty} Q(u)\,du \big\| \leq Ce^{-\sigma t}$, $t \in I$, $\sigma > 2a$, *where* C *is some constant, then the systems* (1) *and* (2) *are asymptotically equivalent.*

The following statement [1] establishes that the estimate $\sigma > 2a$ is unimprovable in a whole set of linear systems (1) with piecewise continuous matrices of coefficients.

Theorem 2. *For any number* $a > 0$ *there exist system* (1) *with piecewise continuous matrix of coefficients with the norm* $\|A(t)\| \leq a$ *for* $t \in I$ *and the piecewise continuous matrix* $Q(t)$, *satisfying the condition* $\big\| \int\limits_t^{+\infty} Q(u)\,du \big\| \leq Ce^{-2at}$, $t \in I$, *such that systems* (1) *and* (2) *are not asymptotically equivalent.*

The following assertion establishes [2] the integral test of asymptotic equivalence of systems (1) and (2).

Theorem 3. *If the matrix of perturbations* $Q(t)$ *of system* (2) *satisfies the condition* $\overline{\lim\limits_{t \to +\infty}} \int\limits_t^{+\infty} \big\| X_A(t,\tau) Q(\tau) X_A(\tau,t) \big\|\, d\tau < 1$, *where* $X_A(t,\tau)$ *is the Cauchy matrix of system* (1), *then system* (2) *is equivalent to system* (1).

2 Coefficients and Exponents of Reducibility of Linear Systems

Let the perturbation $Q(t)$ satisfy the condition

$$\|Q(t)\| \le C(Q)e^{-\sigma t}, \;\; \sigma \ge 0, \;\; t \ge 0, \tag{3}$$

or the more general condition

$$\lambda[Q] \equiv \overline{\lim_{t\to+\infty}} \, t^{-1} \ln \|Q(t)\| \le -\sigma \le 0. \tag{4}$$

These perturbations for $\sigma = 0$ in both cases (3) and (4) we assume additionally to be vanishing at infinity: $Q(t) \to 0$ as $t \to +\infty$. To every system (1) we put into correspondence the sets $R(A)$ and $R_\lambda(A)$ of those values of the parameter σ in (3) and (4) for which perturbed system (2) for any perturbation $Q(t)$ satisfying (3) or, respectively, (4), is asymptotically equivalent to non-perturbed system (1).

Definition. An exact lower bound $r(A)$ of the set $R(A)$ (an exact lower bound $\rho(A)$ of the set $R_\lambda(A)$) will be called a coefficient of reducibility (an exponent of reducibility) of system (1).

Theorem 4 ([2]). *The coefficient of reducibility $r(A)$ and the exponent of reducibility $\rho(A)$ of every system* (1) *with piecewise continuous bounded coefficients coincide.*

This fact allows one to define a new asymptotic invariant of linear systems, i.e., the coefficient of reducibility of the system r_A, as a general value of its coefficient and exponent of reducibility. However, despite the fact that the above-mentioned values coincide for every system (1), the behavior of the coefficient of reducibility r_A is distinct with respect to perturbations (3) and (4). The following theorem [2] establishes this difference.

Theorem 5. *For any number $a > 0$, there exists system* (1) *with the coefficient of reducibility $r_A = 2a$ such that system* (2) *with any piecewise continuous perturbation Q satisfying condition* (3) *with Q is reducible to the initial system* (1) *and non-reducible to that system for some perturbation Q satisfying the condition* (4) *with $\sigma = r_A$.*

Thus it follows from the above results (see also [3, 4]) that both the sets $R(A)$ and $R_\lambda(A)$ are the intervals, and $(2a, +\infty) \subset R_\lambda(A) \subset R(A)$. In addition, despite the fact that the values $r(A)$ and $\rho(A)$ coincide for any system (1), their properties are distinct: there exist systems (1) in which $R(A) = R_\lambda(A) = (r_A, +\infty)$ and, at the same time, there exist systems (1) for which $R(A) = [r_A, +\infty)$ and $R_\lambda(A) = (r_A, +\infty)$. Moreover [5], unlike the coefficient $r(A)$ which can or cannot belong to the set $R(A)$, the exponent of reducibility $\rho(A)$ of system (1) never belongs to the set $R_\lambda(A)$.

Establishment of the above-mentioned properties of the coefficient of reducibility of the linear differential system allows one to investigate certain parametric properties of the so-called sets of non-reducibility $N_r(a, \sigma)$ and $N_\rho(a, \sigma)$ $\sigma \in (0, 2a]$ of all those systems (1) for every of which there exists a non-reducible to it system (2) with the matrix $Q(t)$ satisfying, respectively, either condition (3), or the more general condition (4).

These sets are non-empty for $\sigma \in (0, 2a]$ and empty for $\sigma > 2a$. Moreover, these sets get strictly narrow as parameter $\sigma \in (0, 2a]$ increases and for $\sigma \in (0, 2a]$ they do not coincide with each other.

References

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