Periodic Solution to Two-Dimensional Half-Linear System

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On the interval $[0,\omega]$ we consider the system of differential equations

$$\begin{aligned} u_1' &= p_1(t)|u_2|^{\lambda_1}\operatorname{sgn} u_2 + f_1(t,u_1,u_2),\\ u_2' &= p_2(t)|u_1|^{\lambda_2}\operatorname{sgn} u_1 + f_2(t,u_1,u_2), \end{aligned} \tag{1}$$

subjected to the periodic-type boundary conditions

$$u_1(0)-u_1(\omega)=h_1(u_1,u_2), \quad u_2(0)-u_2(\omega)=h_2(u_1,u_2). \tag{2}$$

Here, $p_i \in L([0,\omega];\mathbb{R})$, $f_i \in \operatorname{Car}([0,\omega]\times\mathbb{R}^2;\mathbb{R})$, $h_i : C([0,\omega];\mathbb{R})\times C([0,\omega];\mathbb{R}) \to \mathbb{R}$ are continuous functionals bounded on every ball, and $\lambda_i > 0$ such that $\lambda_1\lambda_2 = 1$.

Notation 1. Define the following functions

$$q_i(t,\rho) \overset{def}{=} \sup\Big\{\big|f_i(t,x_1,x_2)\big| : \ |x_i|\le\rho, \ |x_{3-i}|\le\rho^{\lambda_{3-i}}\Big\} \ \text{for a. e. } t\in[0,\omega] \ \ (i=1,2),$$

$$\eta_i(\rho) \overset{def}{=} \sup\Big\{|h_i(u_1,u_2)| : \ \|u_i\|_C\le\rho, \ \|u_{3-i}\|_C\le\rho^{\lambda_{3-i}}\Big\} \ \ (i=1,2).$$

Theorem 1. *Let*

$$\lim_{\rho\to+\infty}\int_0^\omega \frac{q_i(s,\rho)}{\rho}\,ds = 0, \qquad \lim_{\rho\to+\infty}\frac{\eta_i(\rho)}{\rho}=0 \ \ (i=1,2). \tag{3}$$

Let, moreover, $\sigma\in\{1,-1\}$ be such that

$$\sigma p_1(t)\ge 0 \ \text{ for a. e. } t\in[0,\omega], \quad p_1\not\equiv 0, \tag{4}$$

and let there exist $\alpha_i\in AC([0,\omega];\mathbb{R})$ $(i=1,2)$ such that

$$\begin{gathered} \alpha_1'(t) = p_1(t)|\alpha_2(t)|^{\lambda_1}\operatorname{sgn}\alpha_2(t) \ \text{ for a. e. } t\in[0,\omega], \quad \alpha_1(0)=\alpha_1(\omega),\\ \alpha_2'(t) \le p_2(t)|\alpha_1(t)|^{\lambda_2}\operatorname{sgn}\alpha_1(t) \ \text{ for a. e. } t\in[0,\omega], \quad \alpha_2(0)\le\alpha_2(\omega),\\ \sigma\alpha_1(t)>0 \ \text{ for } t\in[0,\omega],\\ \operatorname{meas}\Big\{t\in[0,\omega] : \ \alpha_2'(t) < p_2(t)|\alpha_1(t)|^{\lambda_2}\operatorname{sgn}\alpha_1(t)\Big\} + \alpha_2(\omega)-\alpha_2(0) > 0. \end{gathered}$$

Then the problem (1), (2) *has at least one solution.*

Corollary 1. *Let* (3) *and* (4) *be fulfilled with $\sigma\in\{1,-1\}$. Let, moreover,*

$$\int_0^\omega \sigma p_1(s)\,ds\bigg(\int_0^\omega [\sigma p_2(s)]_-\,ds\bigg)^{\lambda_1} < 2^{1+\lambda_1},$$

$$\int_0^\omega [\sigma p_2(s)]_-\,ds < \int_0^\omega [\sigma p_2(s)]_+\,ds\bigg(1-\frac{1}{2^{1+\lambda_1}}\int_0^\omega \sigma p_1(s)\,ds\bigg(\int_0^\omega[\sigma p_2(s)]_-\,ds\bigg)^{\lambda_1}\bigg)^{\lambda_2}.$$

Then the problem (1), (2) *has at least one solution.*

Remark 1. Theorem 1 and Corollary 1 are applicable in the case when $\int\limits_0^\omega \sigma p_2(s)\,ds > 0$. For the case when $\int\limits_0^\omega \sigma p_2(s)\,ds < 0$ one can use the following assertion.

Theorem 2. *Let* (3) *and* (4) *be fulfilled with* $\sigma \in \{1,-1\}$*. Let, moreover,*

$$\int\limits_0^\omega \sigma p_2(s)\,ds < 0, \quad \int\limits_0^\omega \sigma p_1(s)\,ds\bigg(\int\limits_0^\omega [\sigma p_2(s)]_-\,ds\bigg)^{\lambda_1} < 4^{1+\lambda_1}.$$

Then the problem (1), (2) *has at least one solution.*

Sketch of the proofs. According to the general result established in [1] one can see that the following assertion holds:

Proposition 1. *Let* (3) *be fulfilled. If the problem*

$$\begin{aligned} u_1' &= p_1(t)|u_2|^{\lambda_1}\operatorname{sgn} u_2, \\ u_2' &= p_2(t)|u_1|^{\lambda_2}\operatorname{sgn} u_1, \end{aligned} \tag{5}$$

$$u_1(0) - u_1(\omega) = 0, \quad u_2(0) - u_2(\omega) = 0 \tag{6}$$

has only the trivial solution, then the problem (1), (2) *has at least one solution.*

Then the conditions of Theorems 1 and 2 and Corollary 1 are obtained by direct analysing the non-trivial solutions of the problem (5), (6). □

Remark 2. Results obtained are unimprovable in that sense that neither of the strict inequalities established in Corollary 1 and Theorem 2 can be weakened.

Remark 3. When $\lambda_i = 1$, $p_1 \equiv 1$, $h_i \equiv 0$, $f_1 \equiv 0$, $f_2(t,x,y) = f(t)$ for a. e. $t \in [0,\omega]$, $x, y \in \mathbb{R}$ with $f \in L([0,\omega];\mathbb{R})$, then the problem (1), (2) becomes a periodic problem for the second-order linear equation

$$u'' = p_2(t)u + f(t), \quad u(0) = u(\omega), \quad u'(0) = u'(\omega).$$

In this case, Theorems 1 and 2, and Corollary 1 coincide with the results obtained in [2].

References

[1] R. Hakl, Fredholm type theorem for functional-differential equations with positively homogeneous operators. (submitted)

[2] R. Hakl and P. J. Torres, Maximum and antimaximum principles for a second order differential operator with variable coefficients of indefinite sign. *Appl. Math. Comput.* **217** (2011), No. 19, 7599–7611.